Finitary Permutation Groups

Combinatorics Study Group
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"You wonder and you wonder until
you wander out into Infinity,
where - if it is to be found
anywhere - Truth really exists."
Marita Bonner,
On Being Young – A Woman – and Colored
in *The Crisis* Dec 1925.

A finitary permutation group is a natural generalization of a finite permutation group. The structure of a transitive finitary permutation group is surprisingly simple when its degree is infinite. Here we study primitivity, following P. M. Neumann's work in the 1970s. We also study generalized solubility conditions on these groups. These notes arose from lectures aimed at an audience who had seen some basic permutation group theory, but little abstract group theory.

1 Definitions and Constructions

1.1 Introduction

Throughout, Ω is a set (usually but not always infinite) and $\mathrm{Sym}(\Omega)$ is the group of all permutations of Ω. One cannot hope to prove deep group-theoretic theorems about subgroups of $\mathrm{Sym}(\Omega)$ when Ω is infinite; for since any group G embeds as a regular subgroup of $\mathrm{Sym}(G)$, we would be proving theorems about *all* groups. Thus we need to restrict our groups with a finiteness condition. Here we discuss such a condition.

Infinite algebra tends to rely on the Axiom of Choice. I have indicated its use throughout. It would be interesting to see what happens to the theory of finitary permutation groups without such an axiom.

Definition Let $g \in \mathrm{Sym}(\Omega)$. The *support of* g *on* Ω is the set

$$\mathrm{supp}_\Omega(g) = \{\omega \in \Omega : \omega g \neq \omega\},$$

that is, the set of elements of Ω which g does not fix. Clearly $\Omega = \mathrm{supp}_\Omega(g) \sqcup \mathrm{fix}_\Omega(g)$. Call g *finitary* on Ω is $\mathrm{supp}_\Omega(g)$ is finite, or equivalently if $\mathrm{fix}_\Omega(g)$ is cofinite in Ω. That is, if g "fixes most" of Ω.

Example. (1234) and (12)(34)(56) are finitary permutations of the natural numbers $\mathbb{N}$ but $\prod_{n=1}^{\infty}(2n-1, 2n)$ is not.

Lemma 1.1. *Let* $g, h \in \mathrm{Sym}(\Omega)$. *Then*

1. $\mathrm{supp}_\Omega(gh) \subseteq \mathrm{supp}_\Omega(g) \cup \mathrm{supp}_\Omega(h)$,
2. $\mathrm{supp}_\Omega(g^{-1}) = \mathrm{supp}_\Omega(g)$,
3. $\mathrm{supp}_\Omega(g) = \emptyset$ *if and only if* $g = 1$.

Proof. 1. Suppose that $\omega gh \neq \omega$ and $\omega \notin \mathrm{supp}_\Omega(g)$. Then $\omega g = \omega$, so $\omega h = (\omega g)h \neq \omega$. Therefore $\omega \in \mathrm{supp}_\Omega(h)$.

2. This follows since $\omega g^{-1} \neq \omega$ if and only if $\omega \neq \omega g$.

3. The permutation g fixes all the points of Ω if and only if $g = 1$. □

Let $\mathrm{FSym}(\Omega)$ be the set of all finitary permutations. By 1.1, $\mathrm{FSym}(\Omega)$ is a subgroup of $\mathrm{Sym}(\Omega)$, called the *finitary symmetric group* on Ω.

Exercise. Show that $\mathrm{FSym}(\Omega)$ is a normal subgroup of $\mathrm{Sym}(\Omega)$.

Examples.

1. If $|\Omega| = n$ is finite then $\mathrm{FSym}(\Omega) = \mathrm{Sym}(\Omega) \cong S_n$.
2. Let $(\Omega_i)_{i\in I}$ be a family of *finite* sets and for each $i \in I$, let $G_i \leq \mathrm{Sym}(\Omega_i)$. Let G be the direct product $\mathrm{Dr}_{i\in I} G_i$ (that is, the set of all sequences $(g_i)_{i\in I}$ with $g_i \in G_i$ such that all but finitely many of the g_i are 1). Let Ω be the disjoint union $\bigsqcup_{i\in I} \Omega_i$. Then G is a finitary permutation group on Ω, where the components of G act on the corresponding components of Ω.

There are two more fundamental types of finitary permutation groups.

1.2 Generalized Wreath Products

The development we follow here is from Robinson [11]. Let I be a linearly ordered set (for example $\mathbb{N}$ with the natural ordering), let $(\Omega_i)_{i\in I}$ be a family of sets and for each $i \in I$ let H_i be a *transitive* subgroup of $\mathrm{Sym}(\Omega_i)$. Choose an element 1_i from each Ω_i (note that we have used the Axiom of Choice here when I is infinite). Let Ω be the set direct product of the Ω_i with the respect to the elements 1_i, namely the set of all sequences $(\omega_i)_{i\in I}$ with $\omega_i \in \Omega_i$ such that $\omega_i = 1_i$ for all but finitely many $i \in I$. Set $1 = (1_i)_{i\in I}$ (not to be confused with the identity of a group).

Let $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ be elements of Ω. Then for $j \in I$ we write $x \equiv y \bmod j$ if $x_i = y_i$ for all $i > j$.

Let $h \in H_j$, $x = (x_i)_{i\in I} \in \Omega$ and define h_Ω as follows. If $x \equiv 1 \bmod j$, then $(xh_\Omega)_j = x_j h$ and $(xh_\Omega)_i = x_i$ when $i \neq j$. If $x \not\equiv 1 \bmod j$ then $xh_\Omega = x$. This gives a one-to-one homomorphism $H_j \to \mathrm{Sym}(\Omega)$, $h \mapsto h_\Omega$. Let P_j be the image of this homomorphism. The *wreath product* of the family $(H_i)_{i\in I}$ is

$$W = \mathrm{Wr}_{i\in I} H_i = \langle P_i : i \in I \rangle, \leq \mathrm{Sym}(\Omega).$$

Lemma 1.2. *Using the notation above:*

1. *W does not depend on the choice of the elements 1_i.*

2. *W is transitive on Ω.*

3. *Let $I = \mathbb{N}$ with the natural ordering and let each H_i be a finite permutation group. Then*

$$W = \mathrm{Wr}_{i=0}^{\infty} H_i = H_0 \mathrm{\ Wr\ } H_1 \mathrm{\ Wr\ } H_2 \mathrm{\ Wr\ } \ldots \leq \mathrm{FSym}(\Omega).$$

4. *Let $I = \{1 < 2\}$ and let H_1 and H_2 be finite permutation groups. Then $W = H_1 \mathrm{\ Wr\ } H_2$, the wreath product of Peter Cameron's notes.*

Proof. This is left as an exercise (see [11] volume 2, pages 18-19 for some of the details). Note that in 3, if $h \in H_i$ then

$$|\mathrm{supp}_\Omega(h_\Omega)| = |\mathrm{supp}_{\Omega_i}(h)||\Omega_{i-1}| \ldots |\Omega_0|.$$

□

Exercise. Show that if in Lemma 2 part 3 one takes $I = \mathbb{N}$ with the reverse ordering, then $W = \ldots \mathrm{Wr\ } H_2 \mathrm{\ Wr\ } H_1 \mathrm{\ Wr\ } H_0$ is not finitary.

1.3 Alternating Groups

Let $g \in \mathrm{FSym}(\Omega)$. One can write g as a finite product of finite orbits and thus as a finite product of transpositions (xy) where $x \neq y$, for example $(1234) = (12)(13)(14)$.

Lemma/Definition 1.3. *Let $g \in \mathrm{FSym}(\Omega)$.*

1. *If g has a decomposition into m transpositions and a decomposition into n transpositions then $m \equiv n \bmod 2$. Denote this common value modulo 2 by $\sigma(g)$.*

2. *$\sigma : \mathrm{FSym}(\Omega) \to \mathbb{Z}/2\mathbb{Z}$ is a group homomorphism called the* sign *map.*

The second part of 1.3 holds since sums of even integers are even, sums of odd integers are even and the sum of an odd integer and an even integer is even. We call g *odd* if $\sigma(g) = 1$ and *even* if $\sigma(g) = 0$.

The kernel of σ is called the *alternating group* on Ω, denoted $\mathrm{Alt}(\Omega)$. That is,

$$\mathrm{Alt}(\Omega) = \{g \in \mathrm{FSym}(\Omega) : \ g \text{ is even }\}.$$

If $|\Omega| \geq 2$ then $\mathrm{FSym}(\Omega)/\mathrm{Alt}(\Omega) \cong \mathbb{Z}/2\mathbb{Z}$ and for $|\Omega| = 1$, we have $1 = \mathrm{Alt}(\Omega) = \mathrm{FSym}(\Omega) = \mathrm{Sym}(\Omega)$.

Recall that a group G is *simple* if it has precisely two normal subgroups, necessarily G and 1, and $G \neq 1$.

Theorem 1.4 (Galois). *For $|\Omega| \geq 5$, the group $\mathrm{Alt}(\Omega)$ is simple.*

The infinite simple finitary permutation groups are known (see 2.6 below). The Classification of Finite Simple Groups gives us the finite ones.

2 Primitivity and Imprimitivity

Throughout this section G is a *transitive* subgroup of $\mathrm{FSym}(\Omega)$ and Ω is an *infinite* set, unless stated otherwise.

2.1 Primitivity

A *G-congruence* on Ω is an equivalence relation which G preserves. A *G-block* of Ω is a subset $\Gamma \subseteq \Omega$ such that for every $g \in G$ either $\Gamma g = \Gamma$ or $\Gamma g \cap \Gamma = \emptyset$. The equivalence classes of a G-congruence form a family of disjoint G-blocks whose union is Ω. Such a family of blocks is called a *G-system of imprimitivity* of Ω. Conversely, given one G-block Γ, the set of translates $\{\Gamma g : g \in G\}$ is a *G-system of imprimitivity* of Ω, from which one can recover a G-congruence.

The group G is *primitive* if the only G-congruences on Ω are the diagonal one $\{(\omega, \omega) : \omega \in \Omega\}$ and the universal one $\{\Omega\}$. Clearly G is primitive if and only if the only G-blocks on Ω are the singletons and Ω. We say G is *imprimitive* otherwise.

There is not much to say about primitive finitary permutation groups of infinite degree, except that they are rare.

Theorem 2.1 (The Jordan-Wielandt Theorem). *Let G be a primitive subgroup of* $\mathrm{FSym}(\Omega)$*. Then* $G = \mathrm{Alt}(\Omega)$ *or* $\mathrm{FSym}(\Omega)$*.*

For a proof of 2.1, see [15] Satz 9.4.

2.2 Imprimitivity

We now study imprimitive groups. The following work is due to P. M. Neumann in his papers [4, 5]. At around the same time, D. Segal developed similar ideas in [8, 9].

Lemma 2.2. *Let $G \leq \mathrm{FSym}(\Omega)$. Then any proper G-block of Ω is finite.*

Proof. Choose a block $\Gamma \subset \Omega$. Pick $\omega_1 \in \Gamma$ and $\omega_2 \in \Omega \setminus \Gamma$. Since G is transitive, there is $g \in G$ such that $\omega_1 g = \omega_2$. Thus $\Gamma g \cap \Gamma = \emptyset$. Therefore $\Gamma \subseteq \mathrm{supp}_\Omega(g)$, which is finite. □

The following theorem is a modification of 1.4 in Peter's notes.

Theorem 2.3 (The Fundamental Theorem of Imprimitivity). *Let Γ be a proper block of G and Δ be the set of translates $\{\Gamma g : g \in G\}$. Then the permutation group H induced on Δ is finitary. If G_0 is the permutation group induced on Γ by its setwise stabilizer in G, then Ω can be identified with $\Gamma \times \Omega$ so that $G \hookrightarrow G_0 \,\mathrm{Wr}\, H$. Also G_0 is a finite permutation group.*

Proof. Since G is finitary, H acts finitarily on the set of translates Δ. □

There are two cases of imprimitivity to look at.

Case 1. G has a maximal proper[1] block Γ_0. Call such a group G *almost primitive.*

[1]throughout, "proper block" means proper non-singleton block.

The following theorem explains the name “almost primitive”.

Proposition 2.4. *Let G be an almost primitive subgroup of* $\mathrm{FSym}(\Omega)$*. Then G has a quotient H which is isomorphic to one of* $\mathrm{Alt}(\Omega)$ *or* $\mathrm{FSym}(\Omega)$*. More precisely, $G \hookrightarrow G_0 \,\mathrm{Wr}\, H$ where G_0 is a finite permutation group and H is a primitive finitary permutation group of infinite degree.*

Proof. Let Γ be a maximal proper G-block of Ω and G_0, H, Δ be as in 2.3. Let Θ be a H-block of Δ. Let Γ_0 be the union of all elements of Θ. Then Γ_0 is a block of G in Ω. Since each translate of Γ is also a maximal proper G-block of Ω and Γ_0 contains a translate of Γ, it follows that Γ_0 is one translate of Γ or the whole of Ω. Thus Θ is singleton or is Δ. In other words, H is primitive. The rest of the result follows from 2.3. □

Case 2. If G is imprimitive and has no maximal proper block then call G *totally imprimitive.*

Proposition 2.5. *Let G be a totally imprimitive subgroup of* $\mathrm{FSym}(\Omega)$*.*

1. *Let Δ be a finite subset of Ω. Then there is a proper block Γ of G such that $\Delta \subseteq \Gamma$.*
2. $|\Omega| = \aleph_0$

The second part of 2.5 says that any transitive finitary permutation group of uncountable degree is either primitive or almost primitive. We can use Proposition 2.5 to construct arbitrarily large G-blocks of Ω. This is a justification for the name “totally imprimitive”.

Proof. Using the Axiom of Choice, there is a strictly ascending chain $\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \ldots$ of blocks. Put $\Theta = \bigcup_{i=0}^{\infty} \Gamma_i$. Now Θ is a countable union of finite sets and is infinite. Thus $|\Theta| = \aleph_0$. Also Θ is a block. By 2.2, $\Theta = \Omega$ and hence $|\Omega| = \aleph_0$. Since Δ is a finite subset of Ω, there is j such that $\Delta \subseteq \Gamma_j$. Take $\Gamma = \Gamma_j$. □

Given a proper block Γ of a totally imprimitive subgroup G of $\mathrm{FSym}(\Omega)$, we have $G \hookrightarrow G_0 \,\mathrm{Wr}\, H$, where H acts finitarily on the infinite set $\Delta = \{\Gamma g : g \in G\}$ as in 2.3. If H is primitive or Δ has a maximal proper H-block then Ω has a maximal proper G-block, a contradiction. Thus H is totally imprimitive. Inductively, we see that G embeds into a generalized wreath product $G_0 \,\mathrm{Wr}\, G_1 \,\mathrm{Wr}\, G_2 \,\mathrm{Wr} \ldots$ of finite permutation groups G_i.

2.3 A consequence of the Jordan-Wielandt Theorem

By Galois' Theorem 1.4, the infinite alternating groups are simple. These groups are the only infinite simple finitary permutation groups.

Theorem 2.6 (Mihles/Tyškevič). *Let G be an infinite simple finitary permutation group. Then $G = \mathrm{Alt}(\Omega)$ for some infinite set Ω.*

Proof. Let $(\Omega_i)_{i\in I}$ be the orbits of G. The point stabilizer N_i of Ω_i is a normal subgroup of G. Now one of the $N_i \neq G$. By simplicity, $N_i = 1$. Put $\Omega = \Omega_i$. Then G acts transitively and faithfully on Ω.

If G is primitive then by the Jordan-Wielandt theorem we have $G = \mathrm{Alt}(\Omega)$. When G is almost primitive, G has an image containing $\mathrm{Alt}(\Theta)$ as a normal subgroup for some infinite set Θ. By simplicity, $G = \mathrm{Alt}(\Theta)$.

If G is totally imprimitive then choose $1 \neq g \in G$ and put $\Delta = \mathrm{supp}_\Omega(g)$. There is a proper block Γ containing Δ by 2.5. Now for any $x \in G$ we have $(\Gamma x)\, g = \Gamma x$. Thus the intersection S of the stabilizers of the Γx, as x ranges through G, is non-trivial. Also $S \lhd G$. Since Γ is a proper block, $S \neq G$. Thus G cannot be simple. The result follows. □

3 An application

A group G has *finite Prüfer rank n* if every finitely generated subgroup of G can be generated by n elements, and n is the least integer with this property. If no such n exists we say that G has infinite Prüfer rank. Note that if G has finite Prüfer rank then so does its subgroups and images.

Theorem 3.1 ([7]). *Let G be a transitive subgroup of $\mathrm{FSym}(\Omega)$ and suppose that G has finite Prüfer rank. Then the set Ω is finite.*

In order to prove this theorem, it is enough to compute the rank of two types of group.

Lemma 3.2. *Let p be a prime and Ω be an infinite set.*

1. *The rank of $C_p^{(n)}$, the direct product of n copies of C_p, is n.*

2. $\mathrm{FSym}(\Omega)$ *and* $\mathrm{Alt}(\Omega)$ *have infinite rank.*

Proof. 1. Let $G = C_p^{(n)}$. Then the rank of G is simply the dimension of G as a vector space over $\mathbb{F}_p$, which is n.

2. The group $C_p^{(n)}$ embeds into $\mathrm{Sym}(p^n)$ and each $\mathrm{Sym}(p^n)$ embeds into $\mathrm{FSym}(\Omega)$. Thus $\mathrm{FSym}(\Omega)$ has infinite rank. To see that $\mathrm{Alt}(\Omega)$ has infinite rank, note that $\mathrm{Sym}(p^n)$ embeds into $\mathrm{Alt}(p^n + 2)$. □

Exercise. Let $g, h \in \mathrm{Sym}(\Omega)$. Suppose that $\mathrm{supp}_\Omega(g) \cap \mathrm{supp}\,\Omega(h) = \emptyset$. Show that $gh = hg$.

The Proof of 3.1: We suppose that Ω is infinite. By 3.2, $\mathrm{FSym}(\Omega)$ and $\mathrm{Alt}(\Omega)$ have infinite rank. Thus by 2.1 and 2.4, G is totally imprimitive.

Since G is locally finite, we can choose an element $g \in G$ of prime order p. Put $\Delta = \mathrm{supp}_\Omega(g)$. Now Δ is a finite non-empty set. By 2.5, we can choose a G-congruence with blocks $(\Omega_i)_{i \in I}$ such that $\Delta \subseteq \Omega_1$, say.

Using the transitivity of G, there is $x_i \in G$ such that $\Omega_1 x_i = \Omega_i$. Now for every $i \in I$, we have $\mathrm{supp}_\Omega(g^{x_i}) = \Delta x_i \subseteq \Omega_i$. Thus the g^{x_i} commute. Furthermore, it is easy to see that $\langle g^{x_i} | i \in I \rangle \cong C_p^{(I)}$. Now I is an infinite set, so $C_p^{(I)}$ has infinite rank by 3.2. Thus G does not have finite rank. This is a contradiction. Hence Ω must be finite. □

4 The commutator subgroup and soluble groups

4.1 The shifting property

A transitive finitary permutation group G on an infinite set Ω has the property that for any finite subset Δ of Ω, one can find a permutation in G that moves Δ away from itself completely. This result is a form of *Neumann's Lemma*. In order to prove this, we use a result due to B. H. Neumann and two facts which we leave as exercises. For a different proof, see Cameron [1] Theorem 6.2.

Lemma 4.1 (B. H. Neumann [3]). *Let n be a positive integer and let G be a group. Suppose that G is the union of n cosets of subgroups $C_1, C_2, \ldots, C_n$:*

$$G = \bigcup_{i=1}^{n} C_i g_i.$$

Then the index of at least one of these subgroups in G does not exceed n.

Exercise. Let $G \leq \mathrm{Sym}(\Omega)$ and let $\alpha \in \Omega$. The *stabilizer* of α in G is

$$G_\alpha = \{g \in G : \alpha g = \alpha\} .$$

1. Show that G_α is a subgroup of G.
2. Show that $(G : G_\alpha) = |\alpha G|$.
3. Let G be transitive. Prove that for $\alpha, \beta \in \Omega$ the set $\{g \in G : \alpha g = \beta\}$ is a right coset of G_α.

Theorem 4.2 (P. M. Neumann). *Let $G \leq \mathrm{FSym}(\Omega)$, with G transitive and Ω infinite. Suppose that Δ is a finite subset of Ω. Then there is $g \in G$ such that $\Delta g \cap \Delta = \emptyset$.*

Proof. Suppose that $\Delta g \cap \Delta \neq \emptyset$ for every $g \in G$. Then for every $g \in G$ there are elements $\delta_1, \delta_2 \in \Delta$ such that $\delta_1 g = \delta_2$. Thus

$$G = \bigcup_{\delta_1, \delta_2 \in \Delta} \{g \in G : \delta_1 g = \delta_2\} .$$

Now for any $\delta_1, \delta_2 \in \Delta$, the set $\{g \in G : \delta_1 g = \delta_2\}$ is a coset of the stabilizer G_{δ_1}. Since Δ is finite, G is the union of a finite family of cosets of stabilizers G_δ, for $\delta \in \Delta$. By Lemma 4.1, at least one of the G_δ has finite index in G. But then δG is finite, contradicting the transitivity of G on the infinite set Ω. □

Exercise. Some proofs of the Jordan-Wielandt Theorem use the above result 4.2. Suppose that the Jordan-Wielandt Theorem is true by other means and use the results of section 2 to prove 4.2.

4.2 Soluble and Nilpotent Groups

Let G be any group and $x, y \in G$. The *commutator* of x and y is $[x, y] = x^{-1}y^{-1}xy$. Let $H, K \leq G$. The *commutator* of H and K is

$$[H, K] = \langle [h, k] : h \in H, k \in K \rangle .$$

The *(1st) derived subgroup* of G is $G' = G^{(1)} = [G, G]$. For every non-negative integer n there is an *n-th derived subgroup*, $G^{(n)}$, defined inductively as follows. Let $G^{(0)} = G$. Then given $G^{(i)}$, define $G^{(i+1)}$ to be the group $(G^{(i)})' = [G^{(i)}, G^{(i)}]$ for $i \geq 0$.

Note that $G^{(i)}$ is a normal subgroup of G for all $i \geq 0$. Also G' is the smallest, by inclusion, normal subgroup N of G such that G/N is abelian. Thus G is abelian if and only if $G' = 1$. We call G *perfect* if $G' = G$. If $G^{(n)} = 1$ for some integer n then we say that G is *soluble (of derived length* $\leq n$*)*.

Let $Z(G)$ be the *(1st) centre* of G, that is

$$Z(G) = \{x \in G : xg = gx \ \forall g \in G\}.$$

For every nonnegative integer n, there is an *n-th centre* $\zeta_n(G)$ of G defined as follows:

Put $\zeta_0(G) = 1$. Then given $\zeta_i(G)$, define $\zeta_{i+1}(G)$ to be the normal subgroup of G such that

$$\frac{\zeta_{i+1}(G)}{\zeta_i(G)} = Z\left(\frac{G}{\zeta_i(G)}\right).$$

Note that $Z(G) = \zeta_1(G)$ and G is abelian if and only if $\zeta_1(G) = G$. The group G is called *centerless* if $\zeta_1(G) = 1$. If $\zeta_c(G) = G$ for some integer c then G is *nilpotent (of class* $\leq c$*)*.

Clearly, an abelian group is nilpotent and a nilpotent group is soluble. The converses are not true. For Sym(3) is a centerless soluble group and D_8, the dihedral group of order 8, is nilpotent but is not abelian.

Proposition 4.3. *Let G be an abelian transitive subgroup of* $\mathrm{FSym}(\Omega)$*. Then Ω is finite.*

Proof. We may assume that $G \neq 1$. We show that G is regular; that is, if $1 \neq g \in G$ then $\mathrm{fix}_\Omega(g) = \emptyset$. For then $\mathrm{supp}_\Omega(g) = \Omega$ for any $1 \neq g \in G$ and thus Ω is finite.

Suppose that $1 \neq g \in G$ and that $\omega \in \mathrm{fix}_\Omega(g)$. Let $h \in G$. Then

$$(\omega h)g = \omega gh = \omega h.$$

Thus $\omega h \in \mathrm{fix}_\Omega(g)$. Hence $\Omega = \omega G = \mathrm{fix}_\Omega(g)$. But then $g = 1$. Since $g \neq 1$, we have $\mathrm{fix}_\Omega(g) = \emptyset$, as required. □

Exercise. Let G be a transitive subgroup of $\mathrm{FSym}(\Omega)$ where Ω is an infinite set. Show that G is centerless (and hence cannot be nilpotent).

We shall prove 4.3 for soluble groups later.

Proposition 4.4. *Let $G \leq \mathrm{FSym}(\Omega)$ where Ω is infinite. Let $N \lhd G$. If G is transitive and G/N cannot be represented as a finitary permutation group of infinite degree, then N is transitive.*

Proof. The orbits of N on Ω form a G-system of imprimitivity of Ω (that is, they are the equivalence classes of a G-congruence), say $(\Omega_i)_{i\in I}$.

Suppose that N is not transitive. Then each Ω_i is a proper block of Ω and so each Ω_i is finite. Now G permutes these blocks transitively and finitarily (see 2.3) and N stabilizes these blocks. Thus G/N acts transitively and finitarily on the blocks Ω_i. Moreover, since each Ω_i is finite, and Ω is infinite and the union of the Ω_i, it follows that G/N can be represented as a transitive finitary permutation group of infinite degree. This is a contradiction. Thus N is transitive. □

Corollary 4.5. *Let $G \leq \mathrm{FSym}(\Omega)$ where Ω is infinite. If G is transitive then the commutator subgroup G' is transitive.*

Proof. By 4.3, G/G' cannot be represented as a finitary transitive permutation group of infinite degree. By 4.4, G' is transitive. □

We now show that G' is a "relatively large" subgroup of a transitive finitary permutation group G of infinite degree. In particular, G' is the minimal normal transitive subgroup of G.

Theorem 4.6 (P. M. Neumann). *Let $G \leq \mathrm{FSym}(\Omega)$ where Ω is infinite.*

1. *Suppose that $N \lhd G$ with N transitive. Then $G' \leq N$.*

2. *If G is transitive then G' is perfect.*

Proof. 1. Let $g, h \in G$. We show that $[g,h] \in N$. Put $\Delta = \mathrm{supp}_\Omega(g) \cup \mathrm{supp}_\Omega(h)$. By 4.2, there is $x \in N$ such that $\Delta \cap \Delta x = \emptyset$. Now $\mathrm{supp}_\Omega(g^x)$ and $\mathrm{supp}_\Omega(h^x)$ are subsets of Δx and $\mathrm{supp}_\Omega(g^{-1}), \mathrm{supp}_\Omega(h^{-1})$ and $\mathrm{supp}_\Omega(gh)$ are subsets of Δ. Thus g^x and h^x commute with g^{-1}, h^{-1} and gh. Let

$$c = [g,x][h,x][(gh)^{-1},x].$$

Then

$$c = (x^{-1})^g x x^h (x^{-1})^{(gh)^{-1}} x \in N$$

since N is a normal subgroup of G, and

$$\begin{aligned} c &= g^{-1}g^x h^{-1}h^x gh((gh)^{-1})^x \\ &= g^{-1}h^{-1}ghg^x h^x((gh)^{-1})^x \\ &= [g,h](gh(gh)^{-1})^x = [g,h]. \end{aligned}$$

That is, $[g,h] \in N$. Therefore $G' \leq N$.

2. By 4.5, G' and $G'' = (G')'$ are transitive. By part 1, $G' \leq G''$. Therefore $G' = G''$, as required. □

Corollary 4.7 (J. Wiegold [14]). *Let $G \leq \mathrm{FSym}(\Omega)$ where Ω is infinite. If G is transitive, then G is not soluble.*

5 Local conditions

5.1 Definition of Local Properties

The study of soluble subgroups of $\mathrm{FSym}(\Omega)$ when Ω is infinite is not interesting by 4.7. Here we study subgroups of $\mathrm{FSym}(\Omega)$ that have a "local soluble structure" rather than a "global soluble structure".

Let $\mathcal{P}$ be a property of groups. A group G is *locally-$\mathcal{P}$* if every finitely generated subgroup of G has the property $\mathcal{P}$.

Examples.

1. A locally abelian group G is abelian. For if $x, y \in G$ then $\langle x, y\rangle$ is abelian. Thus x and y commute for all $x, y \in G$.

2. Let Ω be any set. Every subgroup of $\mathrm{FSym}(\Omega)$ is locally finite. For if $x_1, x_2, \ldots, x_n \in \mathrm{FSym}(\Omega)$ then the support of any element that is a product of the x_i is contained in the finite set $\bigcup_{i=1}^{n} \mathrm{supp}_\Omega(x_i)$. Thus $\langle x_1, \ldots, x_n\rangle$ can be regarded as a subgroup of $\mathrm{Sym}(\bigcup_{i=1}^{n} \mathrm{supp}_\Omega(x_i))$ and thus is finite.

5.2 Locally Soluble Groups

Locally soluble transitive finitary permutation groups of infinite degree exist. For example, let G be the group $\mathrm{Wr}_{i=2}^{\infty} C_i$ where C_i is the cyclic group of order i. Then G is locally soluble and is a transitive subgroup of $\mathrm{FSym}(\{2, 3, 4, \ldots\})$. Locally soluble finitary permutations groups satisfy a global generalized solubility condition.

Theorem 5.1 (P. M. Neumann). *Let $G \leq \mathrm{FSym}(\Omega)$ and suppose that G is locally soluble. Then G is hyperabelian of height $\leq \omega$. That is, there is a series*

$$1 = G_0 \leq G_1 \leq \ldots \leq G_n \leq \ldots$$

of G with each $G_n \lhd G$, the union $\bigcup_{n\in\mathbb{N}} G_n = G$ and each factor G_{n+1}/G_n is abelian.

In 5.1, ω is the first infinite ordinal. Note that for Ω infinite, neither $\mathrm{Alt}(\Omega)$ nor $\mathrm{FSym}(\Omega)$ can be locally soluble. Thus G has to be totally imprimitive on any infinite orbit Ω. One uses similar results to 2.5 to prove that the permutation group on Ω induced by G is hyperabelian of height $\leq \omega$. If Ω is finite then the induced permutation group of G on Ω is soluble and in particular is hyperabelian of height $< \omega$. Since G embeds into the direct product of the groups induced from G on the orbits of G, it follows that G is hyperabelian of height $\leq \omega$.

5.3 Locally Nilpotent Groups

Let p be a prime. A *p-group* is a group in which every element has order a power of p. A finite p-group is nilpotent and thus a locally finite p-group is locally nilpotent. Note that there are centerless infinite p-groups.

Let $G = \mathrm{Wr}_{(\mathbb{N},<)}C_2$. Now G is a transitive subgroup of $\mathrm{FSym}(\mathbb{N})$. It is a locally finite 2-group and thus is locally nilpotent. In fact, given any prime p and a sequence $(a_n)_{n\in\mathbb{N}}$ of natural numbers, the wreath product $\mathrm{Wr}_{n\in\mathbb{N}}C_{p^{a_n}}$ is a transitive p-subgroup of $\mathrm{FSym}(\mathbb{N})$. Moreover, given two different sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, the groups $\mathrm{Wr}_{n\in\mathbb{N}}C_{p^{a_n}}$ and $\mathrm{Wr}_{n\in\mathbb{N}}C_{p^{b_n}}$ are not isomorphic. The number of sequences of natural numbers if $2^{\aleph_0}$. It follows that there are at least $2^{\aleph_0}$ transitive finitary locally nilpotent p-groups of infinite degree.

The only locally nilpotent finitary transitive permutation groups of infinite degree are p-groups.

Theorem 5.2 (D. A. Suprunenko). *Let G be a transitive subgroup of* $\mathrm{FSym}(\Omega)$ *where Ω is infinite. If G is locally nilpotent, then G is a p-group for some prime p.*

A locally finite, locally nilpotent group G is a direct product of p-groups ([12] 12.1.1). The following lemma finishes the proof of 5.2.

Lemma 5.3. *Let $G = X \times Y \leq$* $\mathrm{FSym}(\Omega)$ *and suppose that G is transitive. If both X and Y are non-trivial, then Ω is a finite set.*

Proof. Let $1 \neq y \in Y$. Put $\Delta = \mathrm{supp}_\Omega(y)$. Then Δ is finite and non-empty. If $x \in X$ then x and y commute, so if $\omega \in \Delta$ then

$$(\omega x)y = \omega yx \neq \omega x$$

and so $\omega x \in \Delta$. Thus $\Delta x = \Delta$. Hence $\Omega = \Delta G = \Delta XY = \Delta Y$. Now Δ is a union of X-orbits and so Ω is a union of finite X-orbits, since $\Omega = \bigcup_{z \in Y} \Delta z$. In exactly the same way, Ω is a union of finite Y-orbits, say $\bigcup_{i \in I} \Omega_i$. Since Δ is finite, there exist $i_1, i_2, \ldots, i_n$ such that $\Delta \subseteq \bigcup_{j=1}^n \Omega_{i_j}$. Therefore $\Omega = \Delta Y = \bigcup_{j=1}^n \Omega_{i_j}$, a finite set. □

We can generalize Suprunenko's theorem.

Lemma 5.4. *Let $G \leq \mathrm{FSym}(\Omega)$ where Ω is infinite and G is transitive. Let p be a prime. If G' is a p-group then G is a p-group.*

Proof. Let $g \in G$. Put $\Delta = \mathrm{supp}_\Omega(g)$. By 4.2, there is $x \in G$ such that $\Delta \cap \Delta x = \emptyset$. Since $\mathrm{supp}_\Omega(g^x) \subseteq \Delta x$, the elements g^x and g commute. Thus $|[g, x]| = |g^x g| = |g|$. The lemma follows. □

Theorem 5.5 ([6]). *Let $G \leq \mathrm{FSym}(\Omega)$ where Ω is infinite and G is transitive. Suppose that G has a locally nilpotent normal subgroup N such that one of the following hold:*

1. *G/N is abelian;*
2. *G/N is soluble;*
3. *G/N is hypercentral;*
4. *G/N satisfies a non-trivial law;*
5. *G/N has finite Prüfer rank.*

Then G is a p-group for some prime p.

Proof. The point here is that in cases 2, 3, 4 and 5, G/N cannot be represented as a transitive finitary permutation group of infinite degree (we discuss some of these conditions in the next section). Hence N is transitive by 4.4. Using 4.6, $G' \leq N$. Thus G/N is abelian and we are in case 1. By 5.2, N is a p-group for some prime p and so G' is a p-group. By Lemma 5.4, G is a p-group. □

6 Further Results

In this section we list some more results which were not given in the lectures.

6.1 Results of Segal and Wiegold

In [9] Segal studies almost primitive groups, there called *Class A* groups. In [8], he generalizes 2.6 as follows:

Theorem 6.1. *Every infinite simple factor of a finitary permutation group is isomorphic to a full alternating group.*

Let G be a group. Define the i-th centre $\zeta_i(G)$ as before but for any ordinal i, taking $\zeta_l(G) = \bigcup_{i<l} \zeta_i(G)$ for limit ordinals l. We say that G is *hypercentral (of central height* $\leq \alpha$*)* if $\zeta_\alpha(G) = G$. In [14], Wiegold proves the following (in addition to 4.7).

Theorem 6.2. *Let* $G \leq \mathrm{FSym}(\Omega)$*. If* G *is hypercentral then* G *has finite orbits in* Ω *and has central height* $\leq \omega$*.*

6.2 More results of P. M. Neumann

Neumann simultaneously generalized results of Giorgetta and Wiegold and proved the following.

Theorem 6.3 ([4]). *If* $G \leq \mathrm{FSym}(\Omega)$ *and* G *satisfies some non-trivial law then all the* G*-orbits of* Ω *are finite. Consequently such a* G *is residually finite and an FC-group.*

In the same paper, he proves 5.1. Similar methods yield the following result.

Theorem 6.4 ([4]). *If there is a finite group which is not isomorphic to a section of* G *then all orbits of* G *are finite or countable (and totally imprimitive). Moreover, there is a series*

$$1 = G_0 \leq G_1 \leq \ldots \leq G_n \leq \ldots$$

where $G_n \lhd G$*, the union* $\bigcup_{n \in \mathbb{N}} G_n = G$ *and each* G_{n+1}/G_n *embeds into a restricted direct power of some finite group.*

6.3 A result of Wehrfritz

Theorem 6.5 ([13]). *Let* $H \lhd G \leq \mathrm{FSym}(\Omega)$. *Suppose that for some positive integer* n *there is a set* X *of generators of* G *such that* $|\,\mathrm{supp}_\Omega(x)| \leq n$ *for every* $x \in X$. *If* H *is a* $\langle \mathbf{P}, \mathbf{L} \rangle\, \mathfrak{A}$*-group then the orbits of* H *have length* $\leq 2n$.

For a detailed description of the local operator $\mathbf{L}$ and the poly operator $\mathbf{P}$, one should refer to Robinson [11]. Here $\mathfrak{A}$ is the class of all abelian groups.

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