Supersolubility and Finitary Groups

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Abstract

In this thesis we are concerned mainly with supersolubility in finitary groups. In Chapter 2, we study the local structure of locally supersoluble finitary groups and extend results by Wehrfritz on the local structure of locally nilpotent finitary groups.

In Chapter 3, we take an excursion from supersolubility. We generalize Platonov's theorems on linear groups of finite Prüfer rank and linear groups satisfying non-trivial laws to certain finitary skew linear groups. We show that there are no transitive finitary permutation groups of infinite degree with finite Prüfer rank. We also show that an irreducible finitary linear group of infinite dimension has infinite Prüfer rank and generates the variety of all groups.

Chapter 4 is about supersolubility in irreducible and transitive finitary groups. We prove that a locally supersoluble group which is either a transitive finitary permutation group on an infinite set or an irreducible finitary skew linear group of infinite dimension, is a p-group for some suitably chosen prime p. In the appendix to Chapter 4, we extend these results.

In Chapter 5, we examine paraheight in finitary groups. Immediately we see that the definition of paraheight is not satisfactory in the study of finitary

groups and we offer several alternative definitions.

The final chapter, Chapter 6, is about generalizations of Engel elements. We wish to produce supersoluble analogues of right and left Engel elements. We give some definitions and prove that these behave similarly to Engel elements in finite groups and certain finitary groups of positive characteristic.

Contents

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Notation and Conventions

Throughout this thesis, we shall adopt the following standardized notation, unless we specify otherwise.

- D is a division ring and F is a field;
- $\overline{F}$ denotes the algebraic closure of the field F;
- V is a left vector space over D and $D^{(n)}$ is the n-dimensional row vector space over D;
- G is a group;
- Ω is a set;
- n is an integer;
- $\mathrm{GL}(V)$ is the group of all D-automorphisms of V and $\mathrm{GL}(n, D)$ is the group of all invertible $n \times n$ matrices over D. Also $\mathrm{M}(n, D)$ is the ring of all $n \times n$ matrices over D;
- $\mathrm{Sym}(\Omega)$ is the group of all permutations on Ω.

Most maps will act from the right. Exceptions to this rule will be determinants and bilinear, Hermitian and quadratic forms which will act from the left. All vector spaces will be left vector spaces. We shall assume the Axiom of Choice. If I is an indexing set, we will usually assume that I is well-ordered.

Let $\mathcal{P}, \mathcal{Q}$ be properties of groups. We say that G is *locally-*$\mathcal{P}$ if every finitely generated subgroup of G is contained in a subgroup enjoying the property $\mathcal{P}$. We call G a $\mathcal{P}$*-by-*$\mathcal{Q}$ group if there is a normal subgroup $N \lhd G$ such that N has the property $\mathcal{P}$ and G/N has the property $\mathcal{Q}$.

Chapter 1

Introduction

Classically, finite group theory was the study of finite permutation groups and infinite group theory was the study of linear groups. In this thesis, we shall study generalizations of these objects, the "finitary groups". This chapter will provide a brief introduction to the theory of finitary groups and will give a review of that part of the basic theory of generalized soluble groups that we need.

1.1 Finitary Permutation Groups

Let Ω be a set. The *finitary symmetric group* on Ω is the group $\mathrm{FSym}(\Omega)$ of permutations on Ω which fix all but finitely many elements of Ω. If g is a permutation on Ω, we define its *support* to be

$$\mathrm{supp}_\Omega(g) = \{\omega \in \Omega : \omega g \neq \omega\}\,.$$

Then

$$\mathrm{FSym}(\Omega) = \{g \in \mathrm{Sym}(\Omega) : |\mathrm{supp}_\Omega(g)| < \infty\}\,.$$

A subgroup G of $\mathrm{FSym}(\Omega)$ is called a *finitary permutation group* on Ω and in this case, G has *degree* $|\Omega|$.

Now any finitary permutation $g \in \mathrm{FSym}(\Omega)$ can be written as a finite product of cycles of finite length. Thus we define the *parity* of g in the same way as for permutations on finite sets – that is, g has even parity if it can be written as a product of an even number of transpositions, and has odd parity otherwise. The *alternating group* on Ω is $\mathrm{Alt}(\Omega)$, the subgroup consisting of all even permutations in $\mathrm{FSym}(\Omega)$. The group $\mathrm{Alt}(\Omega)$ is simple when $|\Omega| \geq 5$.

Let $g_1, g_2, \ldots, g_n \in \mathrm{FSym}(\Omega)$ and let $\Delta = \bigcup_{i=1}^{n} \mathrm{supp}_\Omega(g_i)$. Then $\langle g_1, \ldots, g_n \rangle$ can be regarded as a permutation group on the finite set Δ and so is itself finite. In other words, $\mathrm{FSym}(\Omega)$ is locally finite.

Let $G \leq \mathrm{FSym}(\Omega)$. We call G *transitive* if $\omega G = \Omega$ for any $\omega \in \Omega$. Let G be transitive. A *G-congruence* is an equivalence relation on Ω that the action of G preserves. The equivalence classes of such a relation are called *G-blocks*. There are always two G-congruences; namely, the trivial one $\{(\omega, \omega) : \omega \in \Omega\}$ and the universal one $\Omega \times \Omega$. If these are the only two then we call G *primitive*. Otherwise we call G *imprimitive*.

Given any subset Γ of Ω, we define its *normalizer* in G,

$$N_G(\Gamma) = \{g \in G : \Gamma g = \Gamma\}.$$

1.1.1. Proposition (cf. [2] Theorem 1.8). *Let $G \leq \mathrm{FSym}(\Omega)$ be transitive and suppose that there is a G-congruence with distinct blocks $(\Omega_i)_{i \in I}$ with $|I| > 1$. Then G permutes the blocks finitarily and transitively, and so $G / \bigcap_{i \in I} N_G(\Omega_i)$ is a transitive finitary permutation group on I. Moreover, each block Ω_i is finite and thus when Ω is infinite, I is infinite.*

When Ω is infinite, we know all possible primitive subgroups of $\mathrm{FSym}(\Omega)$.

1.1.2. Theorem (The Jordan-Wielandt Theorem). *Let G be a primitive finitary permutation group of infinite degree. Then G is either* $\mathrm{FSym}(\Omega)$ *or* $\mathrm{Alt}(\Omega)$ *for some infinite set* Ω.

For a proof of this, see [51] Satz 9.4.

In the 1970s, P. M. Neumann studied imprimitive subgroups of $\mathrm{FSym}(\Omega)$ when Ω is infinite. There are two cases, both of which can be dealt with relatively easily.

Let G be a transitive subgroup of $\mathrm{FSym}(\Omega)$ and suppose that Ω is infinite. We call G *almost primitive* if there is a maximal proper non-trivial G-congruence. If G is imprimitive but not almost primitive, then we call G *totally imprimitive.* When dealing with these concepts, we do not need much more that the following proposition.

1.1.3. Proposition (P. M. Neumann). *Let $G \leq$* $\mathrm{FSym}(\Omega)$ *where G is transitive and Ω is infinite.*

1. *If G is almost primitive then G has an image isomorphic to* $\mathrm{Alt}(\Omega)$ *or* $\mathrm{FSym}(\Omega)$.

2. *Suppose that G is totally imprimitive and that Γ is a finite subset of Ω. Then there is a proper G-congruence with Γ contained in one of its G-blocks.*

3. *If G is totally imprimitive then both G and Ω are countably infinite.*

For a general discussion of imprimitivity in finitary permutation groups and for proofs of 1.1.3, see [20] Section 2.

We shall need another result of P. M. Neumann:

1.1.4. Lemma (Neumann's Lemma). *Let G be a transitive subgroup of* $\mathrm{FSym}(\Omega)$ *where Ω is an infinite set. Suppose that Δ is a finite subset of Ω. Then there is $g \in G$ such that*

$$\Delta g \cap \Delta = \emptyset.$$

This is Lemma 2.3 in [21].

The commutator subgroup of a finitary permutation group is very important. We conclude this section with results on this topic.

1.1.5. *Suppose that G is an abelian transitive subgroup of* $\mathrm{FSym}(\Omega)$*. Then Ω is finite.*

Proof. Let $g \in G \setminus \{1\}$. Then $\mathrm{supp}_\Omega(g)$ is non-empty and finite. Let $\omega \in \mathrm{supp}_\Omega(g)$ and let $x \in G$. Then

$$(\omega x)g = \omega g x, \neq \omega x,$$

so $\Omega = \omega G = \mathrm{supp}_\Omega(g)$ is finite. □

1.1.6. Proposition. *Let G be a transitive subgroup of* $\mathrm{FSym}(\Omega)$ *where Ω is infinite. Let $N \lhd G$ be such that G/N cannot be represented*[1] *as a transitive finitary permutation group of infinite degree. Then N is transitive. In particular, by 1.1.5, G' is transitive.*

1.1.7. Theorem (P. M. Neumann). *Let G be a transitive subgroup of* $\mathrm{FSym}(\Omega)$ *where Ω is infinite. Then G' is the unique minimal normal transitive subgroup of G. In particular, G' is perfect.*

[1]In this thesis, "represented" does not necessarily mean "represented faithfully".

For proofs of 1.1.6 and 1.1.7, see [21] Lemma 2.1 and Theorem 1 respectively.

1.2 Finitary Linear Groups

Let V be a (left) vector space over the division ring D and let g be a D-automorphism of V. In the affine general linear group $V \rtimes \mathrm{GL}(V)$ on V, we can consider commutators, centralizers and normalizers as in any group. We shall adopt these notations in our theory. Thus we write $[V, g]$ for $V(g-1)$, $C_V(g)$ for the fixed-point stabilizer of g in V and so on.

Now we define a linear analogue of finitary permutation groups. Here the commutator $[V, g]$ plays the rôle that $\mathrm{supp}_\Omega(g)$ did in the previous section.

The *finitary general skew linear group* on V is

$$\mathrm{FGL}(V) = \{g \in \mathrm{GL}(V) : \dim_D[V, g] < \infty\}\,.$$

A subgroup G of $\mathrm{FGL}(V)$ is called a *finitary skew linear group on V, over D, of dimension* $\dim_D V$. The space V becomes a D-G (bi)module in the obvious way.

There is a D-epimorphism $V \to [V, g], v \mapsto [v, g]$ with kernel $C_V(g)$ for any $g \in \mathrm{GL}(V)$. Thus the automorphism g is finitary, that is $g \in \mathrm{FGL}(V)$, if and only if $C_V(g)$ has finite D-codimension in V.

When $n = \dim_D V$ is finite, $\mathrm{FGL}(V) \cong \mathrm{GL}(n, D)$ and G is called *skew linear*. If D is a field, then we drop the word "skew" from the above. Thus, a linear group is (isomorphic to) a subgroup of $\mathrm{GL}(n, F)$ for some positive integer n and some field F.

Let $g_1, g_2, \ldots, g_n \in \mathrm{FGL}(V)$ and let $G = \langle g_1, \ldots, g_n \rangle$. Then $C_V(G) = \bigcap_{i=1}^n C_V(g_i)$ has finite codimension in V, so there is a finite-dimensional subspace W of V with $C_V(G) \oplus W = V$. Let $U = W + [V,G] = W + \sum_{i=1}^n [V, g_i]$. Then U has finite dimension. Furthermore, if $g \in G$ and $w \in W$ then $wg = w(g-1) + w \in [V,G] + W = U$. Also $[V,G]$ is G-invariant, so U is a D-G submodule of V. Thus $V = U \oplus C$ for U a finite-dimensional D-G submodule and some $C \leq C_V(G)$. Now G can be regarded as a skew linear group on U. Thus $\mathrm{FGL}(V)$ is in a sense locally skew linear. Moreover, if $n = \dim_D U$, one can choose a basis of V so that the coordinatized matrices of the elements of G with respect to this basis lie in

$$\begin{pmatrix} \mathrm{GL}(n,D) & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence when D is a field, we can define the *determinant* of an element $g \in \mathrm{FGL}(V)$, as in the finite-dimensional case.

Sometimes in this thesis, we will have to restrict the division ring D so that we can obtain some reasonable results.

Let F be a subfield of the division ring D. The division ring D is *locally finite-dimensional* over F if given any finite collection $x_1, \ldots, x_n \in D$, the subring $\langle F, x_1, \ldots, x_n \rangle$ is finite-dimensional over F. Note that F does not have to be central in D here and there is a possible ambiguity as to whether we take left dimensions or right dimensions of the subrings $\langle F, x_1, \ldots, x_n \rangle$. However no such ambiguity arises (see, for example, Stewart [36] Lemma 2.1). If D is locally finite-dimensional over F and F is central in D, then D is a locally finite-dimensional F-algebra.

Let $G \leq \mathrm{FGL}(V)$. We say that G is *irreducible* if V is irreducible as a D-G

module, that is if V has precisely two D-G submodules, namely 0 and V. If V is non-zero, but not irreducible, we say that V and G are *reducible*. The group G is *homogeneous* if V is the direct sum of isomorphic D-G irreducible submodules. We call G *completely reducible* if V is the direct sum of D-G irreducible submodules. In this case, we can collect the isomorphic D-G irreducible submodules together into *homogeneous components* of V.

Let G be an irreducible subgroup of $\mathrm{FGL}(V)$. A *G-system of imprimitivity* of V is a family of subspaces $(V_\omega)_{\omega\in\Omega}$ such that $V = \bigoplus_{\omega\in\Omega} V_\omega$ and G preserves this decomposition. That is, for any $\omega_1 \in \Omega$ and $g \in G$ there is $\omega_2 \in \Omega$ such that $V_{\omega_1} g = V_{\omega_2}$. There is always one such system, namely V on its own. If this is the only one, we call G *primitive*. Otherwise we say that G is *imprimitive*.

1.2.1. Proposition. *Let G be an irreducible subgroup of $\mathrm{FGL}(V)$ with proper G-system of imprimitivity $(V_\omega)_{\omega\in\Omega}$. Then:*

1. *each V_ω is irreducible as D-$N_G(V_\omega)$ module;*
2. *G permutes the V_ω finitarily and transitively, and so $G/\bigcap_{\omega\in\Omega} N_G(V_\omega)$ is a transitive finitary permutation group on Ω;*
3. *each V_ω has finite dimension over D and so when $\dim_D V$ is infinite we have $|\Omega| = \dim_D V$. Furthermore $\dim_D V_\omega = \dim_D V_{\omega_0}$ for all $\omega, \omega_0 \in \Omega$;*
4. *the group $\bigcap_{\omega\in\Omega} N_G(V_\omega)$ is a subdirect product of isomorphic skew linear groups of dimension $\dim_D V_\omega$.*

This is essentially 2.2.3 of [23].

The following is a generalization of Clifford's Theorem. The main part on complete reducibility of normal subgroups was proved by Wehrfritz ([47] Proposition 8) and earlier by Meierfrankenfeld ([19] Theorem 5.1) when D is a field. The rest can be proved in a similar manner to the classical result.

1.2.2. Theorem ("Clifford's Theorem"). *Let $G \leq \mathrm{FGL}(V)$ and let $N \lhd G$. If G is completely reducible then so too is N. If G is irreducible, then the D-N homogeneous components form a G-system of imprimitivity of V.*

1.2.3. Proposition. *Let V be an infinite-dimensional vector space over D and let G be an irreducible subgroup of $\mathrm{FGL}(V)$. Let $1 \neq N \lhd G$ be such that G/N cannot be represented as a transitive finitary permutation group of infinite degree. Then N is irreducible. In particular, G' is irreducible.*

Proof. By 1.2.2, N is completely reducible and the distinct D-N homogeneous components $(V_i)_{i \in I}$ form a G-system of imprimitivity of V. The group $\overline{G} = G/\bigcap_{i \in I} N_G(V_i)$ is a transitive subgroup of $\mathrm{FSym}(I)$ where $|I|$ is infinite or 1 by 1.2.1. But N normalizes each V_i, so $\overline{G}$ is an image of G/N. By hypothesis, $|I| = 1$. Therefore N is homogeneous.

Let U be any D-N irreducible submodule of V. Suppose that U has finite dimension. Since U is a non-trivial D-N module, $[U, n]$ has dimension ≥ 1 for some $n \in N$. Now V is a direct sum of infinitely many copies of U, so $[V, n]$ must have infinite dimension. This contradicts finitariness, so U has infinite dimension.

If $g \in G$ then $Ug \cap U \neq 0$, since g is finitary. Now Ug is an irreducible D-N submodule of V, so we have $Ug = U$. It follows that U is a D-G submodule of V and so $U = V$. Hence N is irreducible. ∎

1.2.4. Proposition. *Let G be an irreducible and imprimitive subgroup of* $\mathrm{FGL}(V)$ *where* $\dim_D V$ *is infinite. Then G' is the minimal normal irreducible subgroup of G. Thus G' is perfect.*

This is 6.2 of [49].

Proposition 1.2.4 is not true for primitive groups. Meierfrankenfeld (see [19] Example 8.1) has shown that a free group G of infinite rank is an infinite-dimensional primitive irreducible finitary linear group over the rationals. A free group is residually soluble but not itself soluble, so $G > G' > G'' > \ldots$. However, we have the following result in the primitive case.

1.2.5. *Let G be an irreducible and primitive subgroup of* $\mathrm{FGL}(V)$ *where V is infinite-dimensional. Then every non-trivial ascendant (e.g. subnormal) subgroup of G is irreducible and primitive.*

The proof of 1.2.5 can be found in [48].

Let V be a D-G module and let $\mathcal{A}$ be a totally ordered set. A *D-G series* of V is a set of pairs of D-G submodules $\{(\Lambda_\alpha, V_\alpha) : \alpha \in \mathcal{A}\}$ satisfying the following:

1. $\bigcup_{\alpha\in\mathcal{A}} \Lambda_\alpha \setminus V_\alpha = V \setminus \{0\}$,
2. $\Lambda_\alpha \leq V_\beta$ for every $\alpha < \beta$ where $\alpha, \beta \in \mathcal{A}$,
3. $V_\alpha \leq \Lambda_\alpha$ for every $\alpha \in \mathcal{A}$.

The submodules Λ_α, V_α are referred to as *terms* and the D-G modules Λ_α/V_α *factors*. For general properties of these generalized series, the reader is directed to [30] Section 1.2. One fact that we will use is that if $\mathcal{A} =$

$\{\alpha_1 < \ldots < \alpha_n\}$ is finite then $V_{\alpha_1} = 0$, $\Lambda_{\alpha_n} = V$ and $\Lambda_{\alpha_i} = V_{\alpha_{i+1}}$. In other words, we recover the definition of finite series.

A *D-G composition series* of V is a D-G series whose factors are irreducible. By Zorn's lemma, every D-G module has a composition series.

Let $G \leq \mathrm{FGL}(V)$. We call G *unipotent* if for every $g \in G$, the endomorphism $g - 1$ is nilpotent. We call G a *stability group* if there is a D-G series $\{(\Lambda_\alpha, V_\alpha) : \alpha \in \mathcal{A}\}$ of V such that $[\Lambda_\alpha, G] \leq V_\alpha$ for all $\alpha \in \mathcal{A}$.

1.2.6. Theorem. *Let $G \leq \mathrm{FGL}(V)$.*

1. *There is a unique maximal normal stability subgroup $\mathcal{S}(G)$ of G, which contains every normal stability subgroup of G. Furthermore, $\mathcal{S}(G)$ stabilizes every D-G composition series of V and $G/\mathcal{S}(G)$ has a faithful completely reducible finitary linear representation on the direct sum of the composition factors of any composition series of V.*

2. *If G is completely reducible then $\mathcal{S}(G) = 1$.*

3. *If G is a stability group then it is unipotent.*

4. *If D is locally finite-dimensional over its centre, then G has a unique maximal normal unipotent subgroup $\mathcal{U}(G)$ containing every normal unipotent subgroup of G and furthermore $\mathcal{U}(G) = \mathcal{S}(G)$.*

5. *Let G be unipotent. Then G is a p-group for $\mathrm{char} D = p > 0$ and is torsion-free when $\mathrm{char} D = 0$.*

For proof of 1.2.6, see [44] 2.1 and 2.2.

1.3 Generalized Solubility Conditions

Let G be any group. We define the *upper central series* of G as follows. Let $\zeta_0(G) = 1$. Given $\zeta_i(G)$ for any ordinal i, define $\zeta_{i+1}(G)$ to be the (normal) subgroup of G such that $\zeta_{i+1}(G)/\zeta_i(G)$ is the centre of $G/\zeta_i(G)$. If j is a limit ordinal then put $\zeta_j(G) = \bigcup_{i<j} \zeta_i(G)$. We have an ascending characteristic series of G,

$$1 = \zeta_0(G) \leq \zeta_1(G) \leq \dots .$$

This series becomes stationary. For example if k_0 is an ordinal such that $|k_0| > |G|$ then $\zeta_{k_0}(G) = \zeta_{k_0+1}(G)$. Choose k to be the least k_0 for which this happens. The ordinal k is called the *central height* of G and $\zeta(G) = \zeta_k(G)$ is called the *hypercentre* of G. A *hypercentral*[2] group is a group which coincides with its hypercentre.

Every group G has a unique maximal normal locally nilpotent subgroup $\eta(G)$ called the *Hirsch-Plotkin radical* of G.

Let $N \lhd G$. The subgroup N is called G-*hypercyclic* if there is an ascending series

$$1 = N_0 \leq N_1 \leq \dots \leq N_i \leq \dots \leq N_j = N$$

such that each $N_i \lhd G$ and each factor N_{i+1}/N_i is cyclic. Put

$$\lambda(G) = \langle N \lhd G : N \text{ is } G\text{-hypercyclic } \rangle .$$

This is a characteristic G-hypercyclic subgroup of G. The group G is called *hypercyclic* if it is G-hypercyclic, or equivalently if $\lambda(G) = G$. If G has a finite G-hypercyclic series, then we say that G is *supersoluble*.

[2]Some authors call subgroups of $\zeta(G)$, *hypercentral subgroups* of G. We shall refrain from this.

The following results are well-known:

1.3.1. Proposition. *1. A supersoluble group is nilpotent by finite-abelian.*

2. A hypercentral group is hypercyclic and a hypercyclic group is hypercentral-by-abelian.

3. A locally nilpotent group is locally supersoluble and a locally supersoluble group is locally-nilpotent by abelian.

4. A locally supersoluble, hypercentral by finitely-generated group is hypercyclic.

5. A locally nilpotent, hypercyclic group is hypercentral.

6. A finitely generated hypercyclic group is supersoluble.

Part 1 is [31] 5.4.10 and Part 2 is proved similarly. Part 3 follows from [31] 5.4.6(ii) and from Part 1. Part 4 is [39] Lemma 11.19 and Part 5 can be found in [13] Corollary 1.12 on page 29. Part 6 is due to Baer and a proof can be found in [39] 11.10

1.3.2. Proposition (cf. [31] 5.4.8). *Let G be a supersoluble group and let $N \lhd G$. Then G has a G-supersoluble series (i.e. a finite G-hypercyclic series) with N as a term. Any G-supersoluble series of G can be refined to one whose factors are cyclic of prime or infinite order.*

The next result says that we need to look at generalized soluble finitary groups rather than just soluble finitary groups to get interesting results.

1.3.3. Proposition. *A soluble transitive finitary permutation group has finite degree and a soluble irreducible finitary skew linear group has finite dimension.*

Proof. Let G be a transitive subgroup of $\mathrm{FSym}(\Omega)$ with Ω infinite. Then $G' \neq 1$ by 1.1.5 and G' is perfect by 1.1.7. Therefore G cannot be soluble.

If G is a soluble irreducible subgroup of $\mathrm{FGL}(V)$ and $\dim_D(V)$ is infinite, then by 1.2.3 and induction, the last non-trivial term S of the derived series of G is irreducible. But then S is an abelian irreducible finitary skew linear group of infinite dimension. This cannot exist (the proof is similar to that of 1.1.5). □

When we study locally soluble finitary groups, we can usually reduce to an imprimitive group.

1.3.4. Theorem (Wehrfritz [47]). *Let $G \leq \mathrm{FGL}(V)$ and suppose that G is primitive and irreducible. Let H be a locally soluble normal subgroup of G. If $\dim_D(V)$ is infinite then $H = 1$.*

One other notion that we will use is that of a local system. A set of subgroups $\mathcal{L}$ of G is a *local system* of G, if $G = \bigcup_{L \in \mathcal{L}} L$ and for every pair $L, M \in \mathcal{L}$ there is $N \in \mathcal{L}$ such that $L, M \leq N$.

Chapter 2

Local systems of Locally Supersoluble Groups

2.1 Local structure

In 1960, Garaščuk [7] proved that a locally nilpotent linear group is necessarily hypercentral. Nine years later, Zalesskii [52] examined the skew linear case and managed to get similar results for locally nilpotent skew linear groups over locally finite-dimensional division algebras, but with a restriction on the action of the group. It is still an open question, despite much effort, as to whether a locally nilpotent skew linear group over a locally finite-dimensional division algebra is hypercentral.

In 1971, Wehrfritz ([38] Theorem A) extended the result of Garaščuk and proved that a locally supersoluble linear group is hypercyclic. With similar restrictions to those of Zalesskii, Stewart [35] extended Wehrfritz's result to matrix groups over locally finite-dimensional division algebras.

In this chapter we consider locally supersoluble finitary skew linear groups.

2.1.1. Example. *Let D be any division ring and let G be the McLain group $\mathrm{M}(\mathbb{Q}, D)$. The group G is a locally nilpotent (and thus locally supersoluble) finitary skew linear group over D that has no non-trivial cyclic normal subgroups and, in particular, is centreless.*

The McLain group (with respect to $\mathbb{Q}$) is constructed as follows (see [18] or [44] for proofs). Let V be the left vector space over D with ordered basis $(v_i : i \in \mathbb{Q})$. For any two rationals i and j, let φ_{ij} be the linear map of V given by $v_i \mapsto v_j$ and $v_k \mapsto 0$ for all $k \neq i$. Then the *McLain group* $G = \mathrm{M}(\mathbb{Q}, D)$ is the group

$$G = \langle 1 + d\varphi_{ij} : d \in D, i, j \in \mathbb{Q}, i > j \rangle .$$

Now for $d \neq 0$, the space $V(1 + d\varphi_{ij} - 1) = V\varphi_{ij} = Dv_j$ has finite dimension. Thus $G \leq \mathrm{FGL}(V)$. The group G is a stability group, thus is locally nilpotent. It is characteristically simple.

If G has a non-trivial cyclic normal subgroup then $\lambda(G) \neq 1$. Now $\lambda(G)$ is a characteristic subgroup of G and G is characteristically simple. Thus G is hypercyclic and so is hypercentral by 1.3.1 Part 5. But then $\zeta_1(G) = G$ is abelian, which is a contradiction.

2.1.1 shows that the finitary skew linear situation is quite different from the linear situation. However even though the groups in 2.1.1 are as far away as possible from being hypercyclic, they do have a rich local structure. Wehrfritz proved in [43] and [45] that certain locally nilpotent finitary skew linear groups have local systems of hypercentral normal subgroups. Here we prove the supersoluble analogue of this theorem.

Until Section 2.3, D is either a *field* F or a *locally finite-dimensional division algebra over a perfect field* F unless stated otherwise. When D is a field, we can assume that it is *algebraically closed*; for clearly $\mathrm{FGL}(V)$ embeds into $\mathrm{FGL}(V \otimes_D \overline{D})$ for any field D.

In this setting, any subgroup G of $\mathrm{FGL}(V)$ has a unique maximal normal unipotent subgroup $\mathcal{U}(G)$ by 1.2.6.

Let $g \in \mathrm{FGL}(V)$. We call g a *d-element* if $\overline{F} \otimes_F F[g]$ is semisimple Artinian. Since F is perfect, g is a d-element if and only if V is completely reducible as an $F[g]$-module (see [39] Theorem 1.24).

Any element g of $\mathrm{FGL}(V)$ can be thought of as a skew linear map. Thus there is a unique unipotent element g_u and a unique d-element g_d such that $g = g_u g_d = g_d g_u$ (see [34] page 84). This is the *Jordan decomposition* of g. Also, locally nilpotent subgroups of $\mathrm{FGL}(V)$ have a Jordan decomposition:

2.1.2. Lemma. *Let G be a locally nilpotent subgroup of* $\mathrm{FGL}(V)$. *Then the maps $g \mapsto g_u$ and $g \mapsto g_d$ are homomorphisms of G onto subgroups G_u and G_d (respectively) of* $\mathrm{FGL}(V)$. *Also $[G_u, G_d] = 1$ and $GG_u = GG_d = G_u \times G_d$, the* Jordan decomposition *of G. Furthermore, the kernel of $g \mapsto g_d$ is $\mathcal{U}(G)$.*

Proof. Everything except the last part of the statement is the content of [43] 2.8. Let K be the kernel of the map $G \to G_d$. If $g \in \mathcal{U}(G)$ then $g = g_u$, that is $g_d = 1$ by uniqueness of the decomposition. Thus $g \mapsto 1$ and $\mathcal{U}(G) \leq K$. Conversely, $K = \{g = g_u g_d : g_d = 1\}$, so K is a unipotent normal subgroup of G. Thus $K = \mathcal{U}(G)$. □

2.1.3. Lemma (Wehrfritz). *Let $G \leq$* $\mathrm{FGL}(V)$ *and let N be a locally nilpotent normal subgroup of G with $\mathcal{U}(N) = 1$. For every finite subset X of G,* *there is a normal subgroup K of G with $X \subseteq K$ and $N \cap K$ hypercentral (of central height $\leq \omega 2$).*

2.1.3 is a restatement of 4.3(d) of [45] for D a field and 4.2(d) of [43] otherwise.

2.1.4. Lemma. *Let $G \leq \mathrm{FGL}(V)$ and let X be any subset of G for which $n = \dim_D[V, X]$ is finite. Put $N = \eta(\langle X^G \rangle)$. Then N_u is nilpotent of class $\leq 2n$.*

Proof. Let $\overline{G} = G(N_d \times N_u)$. Pick any D-$\overline{G}$ composition series of V, say $(V_\alpha, \Lambda_\alpha)_{\alpha \in I}$. Intersecting this series with $[V, X]$ and removing repetitions, we obtain a finite series

$$\begin{aligned} 0 = [V, X] \cap V_{\alpha_1} &\leq [V, X] \cap \Lambda_{\alpha_1} = [V, X] \cap V_{\alpha_2} \leq \dots \\ &\leq [V, X] \cap \Lambda_{\alpha_{i-1}} = [V, X] \cap V_{\alpha_i} \leq \dots \\ &\leq [V, X] \cap \Lambda_{\alpha_n} = [V, X] \end{aligned}$$

where $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$ are elements of I.

Consider the series

$$0 \leq V_{\alpha_1} \leq \Lambda_{\alpha_1} \leq V_{\alpha_2} \leq \dots \leq V_{\alpha_n} \leq \Lambda_{\alpha_n} \leq V. \tag{2.1}$$

Now $[V, X] = [V, X] \cap \Lambda_{\alpha_n}$ and $[V_{\alpha_1}, X] \leq [V, X] \cap V_{\alpha_1} = 0$. Also if $1 < i \leq n$ then

$$[V_{\alpha_i}, X] \leq [V, X] \cap V_{\alpha_i} = [V, X] \cap \Lambda_{\alpha_{i-1}} \leq \Lambda_{\alpha_{i-1}}.$$

Furthermore, the series 2.1 is $\overline{G}$-invariant and since $N \leq \left\langle X^{\overline{G}} \right\rangle$, we have $[V, N] \leq \Lambda_{\alpha_n}, [V_{\alpha_1}, N] = 0$ and $[V_{\alpha_i}, N] \leq \Lambda_{\alpha_{i-1}}$ for $1 < i \leq n$.

Let $B \leq A$ be D-$\overline{G}$ modules with $[A, N] \leq B$ and choose $n \in N$. On the factor A/B, we have $1 \equiv n \equiv n_u n_d$ as a Jordan decomposition for n. By the uniqueness of Jordan decomposition, $n_u \equiv 1$ on the factor A/B. In other words, $[A, N_u] \leq B$. Thus $[V, N_u] \leq \Lambda_{\alpha_n}, [V_{\alpha_1}, N_u] = 0$ and $[V_{\alpha_i}, N_u] \leq \Lambda_{\alpha_{i-1}}$ for $1 < i \leq n$.

Put $C_\alpha = C_{\overline{G}}(\Lambda_\alpha / V_\alpha)$. Then $N_u C_\alpha / C_\alpha$ is a unipotent normal subgroup of the irreducible group $\overline{G}/C_\alpha$ for every $\alpha \in I$. By 1.2.6, $N_u \leq C_\alpha$ and so $[\Lambda_\alpha, N_u] \leq V_\alpha$ for every $\alpha \in I$.

Consequently, N_u stabilizes the series 2.1 and thus N_u is nilpotent of class $\leq 2n$ by [15] Theorem 1.C.1. □

We are now in the position to prove one of the main results of this chapter.

2.1.5. Theorem. *Let G be a locally-nilpotent by abelian subgroup of* FGL(V). *Then G has a local system of hypercentral by finitely-generated-abelian normal subgroups.*

Proof. Let X be a finite subset of G and put $H = \langle X^G \rangle$. Set $N = \eta(H) = H \cap \eta(G)$. Since $G/\eta(G)$ is abelian,

$$H/N \cong H\eta(G)/\eta(G) = \langle \eta(G)x : x \in X \rangle .$$

Thus H/N is a finitely generated abelian group. Now it is sufficient to prove that N is hypercentral.

By 2.1.2, there is an epimorphism $N \longrightarrow N_d$ with kernel $U = \mathcal{U}(N)$. Also $U \lhd G$; for U^g is a unipotent normal subgroup of N for every $g \in G$. Let $\overline{X}$ be the set $\{Ux : x \in X\}$. Now N/U is a locally nilpotent normal subgroup of H/U with $\mathcal{U}(N/U) = 1$. Thus by 2.1.3, there is $K \lhd G/U$ with $\overline{X} \subseteq K$ and $(N/U) \cap K$ hypercentral. Also $\left\langle \overline{X}^G \right\rangle = H/U$, so $N_d \cong N/U$ is hypercentral.

By 2.1.4, N_u is nilpotent and thus $N_u N_d$ is hypercentral (of height $\leq \omega 2$). Since $N \leq N_u N_d$, we have that N is hypercentral, as required. ☐

2.1.6. Corollary. *Let G be a locally supersoluble subgroup of* $\mathrm{FGL}(V)$*. Then G has a local system of hypercyclic normal subgroups.*

Proof. By 2.1.5, any finite subset X of G is contained in a hypercentral by finitely-generated normal subgroup $H \lhd G$. Since H is locally supersoluble, H is hypercyclic by 1.3.1 Part 4. ☐

2.2 Monomiality

We now deduce that certain irreducible locally supersoluble groups of infinite dimension must be imprimitive. This is a special case of 1.3.4. Note that 1.3.4 does not need the restrictions on D that we have made here. We also show that irreducible locally supersoluble finitary linear groups over algebraically closed fields are "monomial", so in this case they are as imprimitive as you can get.

2.2.1. Lemma. *An irreducible finitary group G of infinite dimension over an arbitrary division ring E has no non-trivial cyclic normal subgroups. In particular, it is centreless.*

Proof. Let G be an irreducible subgroup of $\mathrm{FGL}(U)$ where U is a vector space over E. Suppose that $1 \neq \langle x \rangle \lhd G$. Now if $g \in G$ then

$$[U, x]g = [Ug, x^g] \subseteq [U, \langle x \rangle] = [U, x].$$

We have shown that $[U, x]$ is a non-zero E-G submodule of U and since U is irreducible, it follows that $U = [U, x]$ has finite dimension. ☐

2.2.2. Proposition. *Let H be a locally-nilpotent by abelian (e.g. locally supersoluble) normal subgroup of $G \leq \mathrm{FGL}(V)$. Suppose that V has infinite dimension. If G is irreducible and primitive then $H = 1$.*

Proof. Let $x \in H$. By 2.1.5, $H_1 = \langle x^H \rangle$ contains a hypercentral normal subgroup H_2 with the quotient H_1/H_2 abelian. By 1.2.5, H_2 and H_1 are irreducible. By 2.2.1, H_2 is centerless, that is $H_2 = 1$. Therefore H_1 is abelian and again by 2.2.1, $H_1 = 1$. Thus $x = 1$ and $H = 1$. □

Let G be an irreducible subgroup of $\mathrm{FGL}(V)$. The group G is called *monomial* if it has a G-system of imprimitivity $(V_\omega)_{\omega \in \Omega}$ such that each V_ω has dimension 1.

2.2.3. Proposition. *Let G be an irreducible abelian by locally-supersoluble subgroup of $\mathrm{FGL}(V)$ where V is a vector space over the algebraically closed field K. Then G is monomial.*

Proof. By 2.2.2, G is imprimitive with G-system of imprimitivity $\bigoplus_{\omega \in \Omega} V_\omega$, say. Fix ω and let $N = N_G(V_\omega)$. By 1.2.1, V_ω is finite-dimensional and irreducible as a K-N module. Thus $N/C_N(V_\omega)$ is an irreducible linear group. By [39] Theorem 1.14, $N/C_N(V_\omega)$ is monomial.

Let $v_1, \ldots, v_m$ be a monomial basis of V_ω; that is $(Kv_i)_{1 \leq i \leq m}$ is a N-system of imprimitivity of V_ω. Now $v_i g \in V_\omega g$ for $1 \leq i \leq m$ and $g \in G$, so it is enough to show that $v_1 g, \ldots, v_m g$ is a monomial basis of $V_\omega g$ under the action of $N_G(V_\omega g)$. For then if T is a right transversal of G to N then $(v_i t : 1 \leq i \leq m, t \in T)$ is a monomial basis for G.

Let $n \in N$. Then $v_i g n = (v_i n^{g^{-1}})g = (\alpha v_j)g$ for some $\alpha \in F$ and $1 \leq j \leq m$, as required. □

Proposition 2.2.3 applies to finitary linear groups over algebraically closed fields, but we can get around this restriction and obtain some results when the ground field is not algebraically closed.

Let $G \leq \mathrm{FGL}(V)$ where V is a vector space over K. We call G *absolutely irreducible* if it is irreducible when regarded as a subgroup of $\mathrm{FGL}(V \otimes_K \overline{K})$. If we are prepared to extend our field slightly, we may assume that an irreducible group is absolutely irreducible.

2.2.4. Theorem (Leinen [16]). *Let $G \leq \mathrm{FGL}(V)$ where V is a vector space over the field K and suppose that G is irreducible. Then there is a field L containing K such that the degree $[L : K]$ is finite, V is a vector space over L and G is absolutely irreducible as a subgroup of $\mathrm{FGL}({}_LV)$. Furthermore $[L : K]$ divides $\dim_K[V, g]$ for every $g \in G$.*

2.2.5. Corollary. *Let G be a locally supersoluble subgroup of $\mathrm{FGL}(V)$, where V is a vector space over the field K.*

1. *If G is irreducible and $\dim_K V$ is infinite then there is an abelian normal subgroup A of G such that G/A is a transitive finitary permutation group of infinite degree.*

2. *If G is completely reducible then G is abelian by locally-finite.*

Proof. Suppose that G is irreducible. By Leinen's Theorem 2.2.4, we may assume that K is algebraically closed. Now by 2.2.3, G is monomial. Applying 1.2.1 Parts 2 and 4, G has an abelian normal subgroup A such that G/A is a transitive finitary permutation group. In particular, G/A is locally-finite.

Also $\dim_K(V)$ is infinite, so the degree of the permutation group G/A is infinite giving Part 1.

If G is completely reducible then it embeds into a direct product of irreducible groups each an image of G. We have seen that any irreducible group of infinite dimension is abelian by locally-finite. The finite-dimensional case is a consequence of [39] Theorem 1.14. The result follows. □

2.3 Modulo the Hirsch-Plotkin radical

We have seen that in general a locally supersoluble group is locally-nilpotent by abelian. We can say more when G is a finitary group over a certain type of division ring. In order to do this, we use the local Zariski topology. We shall also use this topology in the next chapter.

For the rest of this chapter, D is a *division ring that is locally finite-dimensional over a subfield* F and V is a left vector space over D. Some of the ideas in this section are developed for D a field in Puglisi [28].

Let $G \leq \mathrm{FGL}(V)$ and $X \leq Y$ be finitely generated subgroups of G. Now X is skew linear over D, say $X = \langle x_1, \ldots, x_s \rangle \leq \mathrm{GL}(n, D)$ and further, $X \leq \mathrm{GL}(n, E)$ where E is the subring of D generated by F together with the entries of the matrices $x_1, \ldots, x_s$. The ring E is finite-dimensional over F, say of dimension m, and so $X \leq \mathrm{GL}(mn, F)$. Therefore X carries the usual Zariski topology of linear groups (see [39] Chapter 5) and has a connected component X° containing the identity. Using the proof of [36] Proposition 2.2, we see that the topology induced on X from the Zariski topology on Y, coincides with the Zariski topology on X. Thus X° is well-defined and $X^\circ \leq Y^\circ$.

2.3.1. Lemma ([39] Lemma 5.2). *Let G be a linear group. Then G° is* *normal in G and has finite index in G. Moreover G° is contained in all closed subgroups of finite index in G.*

Set

$$G^- = \bigcup\{X^\circ : X \text{ is a finitely generated subgroup of } G\}.$$

The following result is well-known.

2.3.2. Lemma. *Let $G \leq \mathrm{FGL}(V)$ and let $\mathfrak{X}$ be a class*[1] *of groups.*

1. *$G^- \lhd G$ and G/G^- is locally finite.*
2. *If for each finitely generated subgroup X of G, we have $X^\circ \in \mathfrak{X}$ then G^- is locally-$\mathfrak{X}$.*

Proof. Each $X^\circ \lhd X$ and the X° form a local system of G^-. Thus $G^- \lhd G$ and 2 follows. A finitely generated subgroup of G/G^- has the form G^-X/G^- where X is a finitely generated subgroup of G. Now X° has finite index in X by 2.3.1 and $X^\circ \leq G^-$. Thus G^-X/G^- is finite. □

2.3.3. Proposition. *Suppose that G is a locally supersoluble subgroup of* $\mathrm{FGL}(V)$*. Then G is locally-nilpotent by periodic-abelian.*

Proof. Let X be a finitely generated subgroup of G. Then X is supersoluble. By 1.3.1 Part 1, there is a nilpotent normal subgroup N of finite index in X. By [39] 5.11(ii), the Zariski closure $\overline{N}$ of N in X is also nilpotent and trivially of finite index in X. Thus by 2.3.1, $X^\circ \leq \overline{N}$. That is, X° is nilpotent. Now

[1]By a class of groups $\mathfrak{X}$, we mean a class in the usual sense together with the property that (a) $\mathfrak{X}$ contains the trivial group, and (b) any group isomorphic to one in $\mathfrak{X}$ is itself in $\mathfrak{X}$.

G^- is locally nilpotent and G/G^- is periodic by 2.3.2. Since $G^- \leq \eta(G)$ and $G/\eta(G)$ is abelian (by 1.3.1 again), the result follows. $\square$

Chapter 3

Varietal Properties and Rank Restrictions

3.1 Prüfer rank and Varieties

A group G is said to have *finite Prüfer rank* r if every finitely generated subgroup of G can be generated by r elements, and if r is the smallest integer with this property. If no such integer r exists, we say that G has *infinite Prüfer rank*. In this chapter we shall use the word "rank" without further qualification to mean "Prüfer rank".

V. Platonov proved the following result in [27]:

3.1.1. Theorem (Platonov's Rank Theorem). *Let G be a subgroup of $\mathrm{GL}(n, F)$ where F is a field of characteristic $p \geq 0$. Suppose that G has finite rank r. Then G is soluble-by-finite. Furthermore, if $p > 0$ then G has an abelian normal subgroup of finite index bounded in terms of r, n and p.*

The notion of rank has the following properties:

3.1.2. *Let $H \leq G$ and $N \lhd G$. If G has finite rank, then so do H and G/N. If G/N and N have finite rank, then so does G.*

In particular, any polycyclic group has finite rank.

A set of *words* W is a subset of the free group on the countably infinite set $\{x_1, x_2, x_3, \ldots\}$ where $x_i \neq x_j$ for positive integers $i \neq j$. Let G be a group and W be any set of words. If $w = x_{i_1}^{j_1} \cdots x_{i_r}^{j_r} \in W$ and $g_1, \ldots, g_r \in G$ then the *value* of w at $(g_1, \ldots, g_r)$ is $w(g_1, \ldots, g_r) = g_{i_1}^{j_1} \cdots g_{i_r}^{j_r}$. If $w(g_1, \ldots, g_r) = 1$ for all $g_1, \ldots, g_r \in G$ then G *satisfies the word* w and G satisfies the *law* $x_{i_1}^{j_1} \cdots x_{i_r}^{j_r} = 1$. The *variety given* by W is the class $\mathfrak{B}(W)$ of all groups G such that all the words in W are satisfied by G. A class of groups $\mathfrak{X}$ is called a *variety* if $\mathfrak{X} = \mathfrak{B}(W)$ for some set of words W. Given any group G, the *variety generated by* G is the smallest variety $\mathfrak{V}(G)$ containing G.

3.1.3. Proposition. *A variety is closed with respect to forming subgroups and quotients, and is residually and locally closed.*

This result is 2.3.3 and 2.3.4 of [31].

In [26], Platonov proves the following theorem, which has a similar flavour to 3.1.1.

3.1.4. Theorem (Platonov's Variety Theorem). *Let G be a linear group. Then either G generates the variety of all groups or G is soluble-by-finite. Consequently, a linear group satisfies a non-trivial law if and only if it is soluble-by-finite.*

P. M. Neumann [20] has proved the next result regarding varieties.

3.1.5. Theorem. *Let $G \leq \mathrm{FSym}(\Omega)$. If G satisfies a non-trivial law then the orbits of G on Ω are finite. Consequently, a transitive finitary permutation group of infinite degree generates the variety of all groups.*

In this chapter, we provide generalizations of 3.1.1 and 3.1.4, and a finitary linear analogue of 3.1.5. We shall prove the following:

- A finitary permutation group with finite rank has finite orbits (Theorem 3.4.1);

- An irreducible finitary skew linear group of infinite dimension over a division ring which is locally finite-dimensional over a subfield, has infinite rank and generates the variety of all groups (Theorem 3.4.2).

3.2 Generalizations of Platonov's Theorems

3.2.1. Theorem. *Let D be a division ring which is locally finite-dimensional over a subfield F, let V be a vector space over D and let $G \leq \mathrm{FGL}(V)$.*

1. *Suppose that G has finite rank. Then G is locally-soluble by locally-finite. Furthermore, if $\mathrm{char}D$ is positive then G is abelian by locally-finite.*

2. *The group G either generates the variety of all groups or is locally-soluble by locally-finite.*

The unitriangular group $G = \mathrm{Tr}_1(3, \mathbb{Z})$ of lower triangular 3×3 integral matrices with 1's on the diagonal, is polycyclic and so has finite rank by 3.1.2. Now G is not abelian by locally-finite, so in general it is not true that a finitary linear group of characteristic 0 with finite rank is abelian by locally-finite – i.e. the restriction of the characteristic in 3.2.1 Part 1 is necessary.

We now head towards a proof of 3.2.1.

3.2.2. Proposition. *Let $G \leq \mathrm{FGL}(V)$ where V is a vector space over D and D is locally finite-dimensional over the subfield F.*

1. *Suppose that $\mathfrak{X}$ is a subgroup-closed class of groups such that if L is linear over a field and $L \geq P \in \mathfrak{X}$, then the Zariski closure of P in L lies in $\mathfrak{X}$. If G is locally ($\mathfrak{X}$-by-finite) then G is locally-$\mathfrak{X}$ by locally-finite.*

2. *Suppose that $\mathfrak{X}$ is a variety. If G is locally ($\mathfrak{X}$-by-finite) then G is $\mathfrak{X}$ by locally-finite.*

Part 2 of 3.2.2 for D is field is Theorem 2.3 of Puglisi [28].

Proof. Let H be a finitely generated subgroup of G. Now H is linear over F (see section 2.3). By assumption there is a normal subgroup $N \in \mathfrak{X}$ with H/N finite.

In case 1, the closure $\overline{N}$ of N in H lies in $\mathfrak{X}$. Now $N \leq \overline{N}$, so $\overline{N}$ is a closed subgroup of finite index in H. Thus $H^\circ \leq \overline{N}$ by Lemma 2.3.1. Since $\mathfrak{X}$ is subgroup closed, we have $H^\circ \in \mathfrak{X}$.

In case 2, H has a closed normal $\mathfrak{X}$-subgroup M of finite index in H by [39] Lemma 10.7 (for example, in the notation there take $M = H \cap \mathcal{A}_F(N)$). By 2.3.1 again, $H^\circ \leq M$. Using 3.1.3, we get $H^\circ \in \mathfrak{X}$.

In either case, we have $H^\circ \in \mathfrak{X}$. By Lemma 2.3.2, G^- is locally-$\mathfrak{X}$ and G/G^- is locally finite. If $\mathfrak{X}$ is a variety, we can apply 3.1.3 to get $G^- \in \mathfrak{X}$. $\square$

Proof of 3.2.1. Let G have finite rank and let H be a finitely generated subgroup of G. Now H is linear over F and has finite rank (by 3.1.3). By Platonov's Rank Theorem 3.1.1, H is soluble-by-finite and when $\text{char}F = \text{char}D > 0$, we have that H is abelian-by-finite. In the positive characteristic case we can apply Proposition 3.2.2 Part 2; the abelian groups form a variety. In general, G satisfies the hypotheses of 3.2.2 Part 1. For in H, the closure of a soluble normal subgroup is a soluble normal subgroup by [39] Theorem 5.11(i). Therefore Part 1 of the Theorem follows.

Suppose that G does not generate the variety of all groups. Then nor does any subgroup either. Let H be any finitely generated subgroup of G. Since H is linear, we can apply Platonov's Variety Theorem 3.1.4. Thus H is soluble-by-finite. An application of 3.2.2 Part 1 finishes the proof. □

3.2.3. Remark (Tits' Alternative). *Let $G \leq \text{FGL}(V)$ where V is a vector space over D and D is a division ring which is locally finite-dimensional over a subfield. Then G is either locally-soluble by locally-finite or contains a free group of free rank 2.*

This result, for D a field, is proved by Phillips ([24] Section 5.6) and also by Puglisi ([28] Theorem 2.1). The proof is an application of 3.2.2 Part 1 together with the Tits' Alternative ([39] Theorem 10.16) for linear groups.

A. Lichtman (see [34] Theorem 1.4.9) has constructed a finitely generated skew linear group G which satisfies a non-trivial law, contains no non-cyclic free subgroup and is not soluble-by-finite. Because G is finitely generated, it cannot be locally-soluble by locally-finite. Thus 3.2.1 Part 2 and 3.2.3 are not true when D is an arbitrary division ring.

3.3 Classification of Simple Locally Finite Finitary Groups

Using the Classification of Finite Simple Groups (CFSG), J. Hall [10] classified all simple locally finite finitary linear groups of infinite dimension. Recently, Wehrfritz [49] finished the classification of all simple locally finite finitary groups and extended some work on primitive groups by Phillips [22]. One can use these results as machinery to deal with locally finite primitive finitary skew linear groups (provided, of course, that we accept CFSG).

3.3.1. Theorem. *Let G be a locally finite primitive finitary skew linear group of infinite dimension. Then G contains a simple normal subgroup S.*

This theorem appears in [22] Phillips for finitary linear groups and as stated in [49] Wehrfritz. Moreover, S turns out to be G'. This was proved for finitary linear groups by Leinen and Puglisi ([17] Theorem B) and independently by Redford in [29]. Wehrfritz proves the general case in [49]. As indicated above, we know all possibilities for S. We describe these now.

Let U be an infinite-dimensional vector space over the field F. Let f be a symplectic form[1] on U, that is an alternating bilinear form on U. The *finitary symplectic group on U with respect to f* is

$$\mathrm{FSp}(U,f) = \{g \in \mathrm{FGL}(U) : f(xg,yg) = f(x,y)\, \forall x,y \in U\},$$

i.e. all of the finitary isometries of f.

[1] We shall only consider non-degenerate forms.

Similarly, given a unitary form f on U, i.e. a Hermitian form with respect to some involution of the field F, we have the *finitary unitary group* $\mathrm{FU}(U, f)$ of all finitary isometries of f. Let $\mathrm{SU}(U, f)$ be the subgroup of all elements in $\mathrm{FU}(U, f)$ of determinant 1.

If q is a quadratic form on U then we have the *finitary orthogonal group* $FO(U, q)$ of all finitary isometries of q, i.e. all $g \in \mathrm{FGL}(U)$ with $q(xg) = q(x)$ for all $x \in U$. We let $\Omega(U, q) = (\mathrm{FO}(U, q))'$.

A *transvection* of U is a map $t_{\varphi,w} : U \to U$, $x \mapsto x + (x\varphi)w$ where w is a fixed vector in U and $\varphi \in U^*$, the dual of U, such that $w\varphi = 0$. Let W be any F-subspace of U^* such that

$$\mathrm{ann}_U(W) = \{u \in U : u\varphi = 0 \, \forall \varphi \in W\} = 0.$$

Then we have the *special transvection* group

$$T(W, U) = \langle t_{\varphi,x} : \varphi \in W, x \in U, x\varphi = 0 \rangle, \leq \mathrm{FGL}(U).$$

3.3.2. Theorem (Hall/Wehrfritz). *Let G be a locally finite simple subgroup of* $\mathrm{FGL}(V)$ *that is not a skew linear group*[2]*, where V is a vector space over the division ring D. Then G is isomorphic to one of the following:*

1. *an infinite alternating group (this is the only possibility when* $\mathrm{char} D = 0$*);*
2. $\mathrm{FSp}(U, f)$ *for some symplectic form f;*
3. $\mathrm{SU}(U, f)$ *for some Hermitian form f;*
4. $\Omega(U, q)$ *for some quadratic form q;*

[2]i.e. not isomorphic to a subgroup of $\mathrm{GL}(n, E)$ for any n and any division ring E

5. $T(W,U)$ *where* $W \leq U^*$ *and* $\mathrm{ann}_U(W) = 0$.

Here U *is some vector space over some subfield* F *of* $\overline{\mathbb{F}_p}$*, the algebraic closure of the field of* p *elements, where* $p = \mathrm{char}D$*. In particular, the subgroup* S *in 3.3.1 is one of the above groups.*

3.3.3. Corollary. *A primitive locally finite infinite-dimensional finitary group* G *contains copies of all finite groups.*

Proof. By 3.3.1 we may assume that G is simple. We apply 3.3.2. An infinite alternating group G contains all finite groups; for $\mathrm{Alt}(n+2)$ contains all finite groups of order n and G contains $\mathrm{Alt}(n)$ for all n.

Turning to the "classical finitary groups", we shall restrict the relevant form to finite-dimensional or countably infinite dimensional subspaces of V and embed all of the symmetric or alternating groups into G.

Symplectic Case: Any symplectic form of dimension $2n$ is equivalent to the one represented by the $2n \times 2n$ matrix z with blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the diagonal and zeros elsewhere. Given a permutation matrix $g = (g_{ij}) \in \mathrm{GL}(n,F)$, enlarge it to a $2n \times 2n$ matrix $g\theta$ consisting of 2×2 blocks (h_{ij}) where $h_{ij} = 1_2$ if $g_{ij} = 1$ and $h_{ij} = 0_2$ if otherwise. Then θ embeds the permutation matrices of $\mathrm{GL}(n,F)$ into $\mathrm{GL}(2n,F)$ and also $(g\theta)^T z(g\theta) = z$. Thus $\mathrm{Sym}(n)$ embeds into $\mathrm{Sp}(2n,F)$ and $\mathrm{FSp}(V,f)$ contains all finite groups.

Orthogonal Case: There is only one type of othogonal group of odd dimension. If F has characteristic 2, then since it is perfect (it is a subfield of $\overline{\mathbb{F}_2}$), we have $\mathrm{O}(2n+1,F) \cong \mathrm{Sp}(2n,F)$ (see Carter [3] Chapter 1), so that $\mathrm{Sym}(n)$ embeds into $\mathrm{O}(2n+1,F)$ by the symplectic case.

Suppose that F has odd characteristic. Any orthogonal form of odd dimension $2n+1$ is represented by the matrix

$$z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_n \\ 0 & 1_n & 0 \end{pmatrix}.$$

There is an embedding $\mathrm{GL}(n,F) \to \mathrm{O}(2n+1,F)$,

$$g \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & (g^{-1})^T \end{pmatrix}$$

and so $\mathrm{Sym}(n)$ embeds into $\mathrm{O}(2n+1,F)$. Thus $\mathrm{Sym}(n)$ embeds into $\mathrm{FO}(V,f)$ for all n.

Therefore, regardless of characteristic, $\mathrm{Alt}(n)$ embeds into $\Omega(V,f)$ for all n, as required.

Unitary Case: We may assume that V has countably infinite dimension. Then any Hermitian form (with respect to a field involution ι) is equivalent to the one represented by the identity. In particular, $\mathrm{SU}(V,f)$ contains copies of all the groups

$$\mathrm{SU}(n,F) = \left\{x \in \mathrm{SL}(n,F) : x^\dagger x = 1\right\},$$

where $x^\dagger = (x\iota)^T$. Now the embedding $\mathrm{GL}(n,F) \to \mathrm{SL}(n+1,F)$,

$$x \mapsto \begin{pmatrix} \frac{1}{\det x} & 0 \\ 0 & x \end{pmatrix}$$

takes $\mathrm{U}(n,F) = \left\{x \in \mathrm{GL}(n,F) : x^\dagger x = 1\right\}$ into $\mathrm{SU}(n+1,F)$. For

$$\begin{pmatrix} \frac{1}{\det x} & 0 \\ 0 & x \end{pmatrix}^\dagger \begin{pmatrix} \frac{1}{\det x} & 0 \\ 0 & x \end{pmatrix} = 1,$$

since in this case $x^\dagger x = 1$ and $(\det x)\iota\det x = \det((x\iota)x) = 1$.

Now permutation matrices satisfy $x^\dagger x = 1$, so $\mathrm{Sym}(n)$ embeds into $\mathrm{SU}(n+1, F)$. Therefore $\mathrm{SU}(V, f)$ contains copies of all finite groups.

Transvection Case: Let $G = T(W, V)$ be a transvection group with $W \leq V^*$ such that $\mathrm{ann}_V(W) = 0$.

Suppose that $v_1, \ldots, v_n \in V$ are linear independent vectors and $\varphi_1, \ldots, \varphi_n \in W$ such that $v_i\varphi_i = 1$ and $v_j\varphi_i = 0$ for $j \neq i$, and further that

$$V = \bigoplus_{i=1}^{n} Kv_i \oplus \bigcap_{i=1}^{n} \ker \varphi_i.$$

Choose $0 \neq v_{n+1} \in \bigcap_{i=1}^{n} \ker \varphi_i$. There is $\varphi \in W$ such that $v_{n+1}\varphi = 1$. Set

$$\varphi_{n+1} = \varphi - \sum_{i=1}^{n} (v_i\varphi)\varphi_i, \in W.$$

Then $v_{n+1}\varphi_{n+1} = v_{n+1}\varphi = 1$ and $v_j\varphi_{n+1} = 0$ for $1 \leq j \leq n$. Also

$$V = \bigoplus_{i=1}^{n+1} Kv_i \oplus \bigcap_{i=1}^{n+1} \ker \varphi_i.$$

This gives a recipe for constructing a copy of $\mathrm{SL}(n, K)$ in G. Fix n and construct v_i and φ_i as above for $1 \leq i \leq n$. Extend $v_1, \ldots, v_n$ to a basis $(v_i)_{i \geq 1}$ of V with $v_j \in \bigcap_{i=1}^{n} \ker \varphi_i$ for $j > n$. Let $t_{\alpha,ij}$ be the transvection $v \mapsto v + \alpha(v\varphi_i)v_j$ where $\alpha \in K$. Then

$$v_k t_{\alpha,ij} = \begin{cases} v_i + \alpha v_j & \text{if } k = i \\ v_k & \text{if } k \neq i \end{cases}$$

So $t_{\alpha,ij}$ can be realized as the $n \times n$ matrix $1_n + \alpha E_{ij}$, where E_{ij} is the $n \times n$ matrix with (i, j)-entry α and all other entries are 0. Thus G contains a subgroup isomorphic to $\mathrm{SL}(n, K)$, for every n. Since $\mathrm{Sym}(n)$ embeds into $\mathrm{SL}(n + 1, K)$, it follows that G contains copies of all finite groups. □

3.4 Irreducible and Transitive Finitary Groups

3.4.1. Theorem. *Let G be a transitive subgroup of* $\mathrm{FSym}(\Omega)$*. If G has finite rank, then Ω is finite. Consequently any subgroup of* $\mathrm{FSym}(\Omega)$ *with finite rank has finite orbits.*

There are two key examples. Let $C_p^{(n)}$ be the direct product of n copies of the cyclic group C_p where p is a prime. Then the rank of $C_p^{(n)}$ is simply its dimension as an $\mathbb{F}_p$-vector space, and this is n.

Let Ω be an infinite set. The group $\mathrm{Alt}(\Omega)$ contains copies of all finite groups, thus it contains $C_p^{(n)}$ for all n and so it must have infinite rank.

Proof. Suppose that Ω is infinite. By 1.1.2 and 1.1.3, any primitive or almost primitive group cannot have finite rank. For such a group has an image containing an infinite alternating group. Thus G is totally imprimitive.

Since G is locally finite, we can choose $g \in G$ with prime order p. Put $\Delta = \mathrm{supp}_\Omega(g)$, a finite non-empty set. By 1.1.3 Part 2, we can choose a G-congruence with blocks $(\Omega_i)_{i \in I}$ such that $\Delta \subseteq \Omega_1$, say. Using the transitivity of G, there is $x_i \in G$ with $\Omega_1 x_i = \Omega_i$. Now for every $i \in I$, we have $\mathrm{supp}_\Omega(g^{x_i}) = \Delta x_i \subseteq \Omega_i$. Thus the supports of the distinct g^{x_i} are disjoint and so the g^{x_i} commute. Furthermore, $\langle g^{x_i} : i \in I \rangle \cong C_p^{(I)}$. Now I is infinite, thus by the first example preceding this proof, $C_p^{(I)}$ has infinite rank. Therefore so does G.

If G is any subgroup of $\mathrm{FSym}(\Omega)$ with finite rank and Γ is an orbit, then $G/C_G(\Gamma)$ is a transitive finitary permutation group on Γ. Hence Γ is finite by the above argument. □

3.4.2. Theorem. *Let G be an irreducible subgroup of* $\mathrm{FGL}(V)$ *where V is an infinite-dimensional vector space over D and D is locally finite-dimensional over a subfield. Then*

1. *G has infinite rank, and*

2. *G generates the variety of all groups.*

Proof. If G is imprimitive then it has an image isomorphic to a finitary transitive permutation group of infinite degree (1.1.1). This image has infinite rank by 3.4.1 and generates the variety of all groups by Neumann's Theorem 3.1.5. Thus so does G.

So we assume that G is primitive. If 1 or 2 does not hold, then by 3.2.1, we see that G is locally-soluble by locally-finite. Hence by 1.3.4, G is locally finite. Now applying 3.3.3, the group G contains copies of all finite groups. But then G must have infinite rank. Also the variety $\mathfrak{V}(G)$ must contain all finite groups and hence all residually finite groups by 3.1.3. The free groups are residually finite (see, for example, [31] 6.1.9). Thus $\mathfrak{V}(G)$ is the variety of all groups. These contradictions finish the proof. $\square$

Chapter 4

Irreducible and Transitive Locally Supersoluble Groups

4.1 Results

We saw in Chapter 3 that irreducibility is a particularly strong condition on certain infinite-dimensional finitary groups; for example, there we proved that such an irreducible group necessarily generates the variety of all groups. In this chapter we look at the effect of irreducibility and transitivity on locally supersoluble finitary groups of infinite degree, extending results on locally nilpotent groups. In [37], Suprunenko shows that a locally nilpotent transitive finitary permutation group of infinite degree is a p-group for some prime p. In [46], Wehrfritz proves the corresponding result for finitary skew linear groups; namely that a locally nilpotent irreducible finitary skew linear group of infinite dimension is a locally finite p-group for some prime p not equal to the characteristic of the scalar division ring. We shall extend these results and prove:

- A locally supersoluble transitive finitary permutation group of infinite degree is a p-group for some prime p (Theorem 4.1.2).
- A locally supersoluble irreducible finitary skew linear group of infinite dimension over the division ring D is a locally finite p-group for some prime $p \neq \mathrm{char}D$ (Theorem 4.1.3).

In either case, the groups in question are locally nilpotent.

We need to know when a finite wreath product is supersoluble. Necessary and sufficient conditions for this were found by Durbin ([6] main theorem).

4.1.1. Theorem (Durbin). *Let A and B be finite nontrivial groups and let $G = A \wr B$. Then G is supersoluble if and only if*

1. *A is nilpotent,*
2. *B is abelian or A and B' are nontrivial p-groups for some prime p, and*
3. *for each prime q dividing the order of A, the exponent of $B/O_q(B)$ divides $q-1$, where $O_q(B)$ is the unique maximal normal q-subgroup of B.*

4.1.2. Theorem. *Let Ω be an infinite set and let G be a locally supersoluble transitive subgroup of* $\mathrm{FSym}(\Omega)$. *Then G is a p-group for some prime p.*

Proof. By 1.3.1, G' is locally nilpotent. Thus $G' \leq \eta(G)$. Now by 1.1.6, $N = \eta(G)$ is transitive. Hence by Suprunenko's result ([37] Theorem 1) N is a p-group for some prime p. Since G is locally finite, we have $N = O_p(G)$.

Suppose for a contradiction that $N \neq G$. Pick $x \in G$ of prime order $q \neq p$

Case 1: $q \not| p-1$. Let X be a finitely generated subgroup of G. Now X is contained in a subgroup Y of G generated by a finite number of elements of N, and x. Since Y is supersoluble, $Y \cap N$ has a Y-supersoluble series whose factors are C_p using 1.3.2. Now x acts as an automorphism on these factors. Since the order of $\mathrm{Aut} C_p$ is $p-1$ and is not divisible by q, it follows that x centralizes each factor in this series. Since $Y = (Y \cap N) \rtimes \langle x \rangle$, it follows that Y is nilpotent and so is X. Thus G is locally nilpotent, a contradiction.

Case 2: $q | p-1$. It is sufficient to assume that G/N has order q. Let $Q = \langle x \rangle$. Then $Q \cap N = 1$, so Q complements N in G, i.e. $G = N \rtimes Q$.

By 1.1.2 a primitive finitary permutation group of infinite degree contains an infinite alternating group. By 1.1.3 an almost primitive finitary permutation group has an image containing an infinite alternating group. Thus primitive and almost primitive groups cannot be locally supersoluble. Therefore G is totally imprimitive.

Set $\Delta = \mathrm{supp}_\Omega(Q) = \mathrm{supp}_\Omega(x)$. By 1.1.3 there is a G-congruence of Ω with blocks $(\Omega_i)_{i \in I}$ such that $\Delta \subseteq \Omega_1$, say. Let $T = \bigcap_{i \in I} N_G(\Omega_i)$. Applying 1.1.1, the set I is infinite and G/T is a transitive subgroup of $\mathrm{FSym}(I)$. Since $\Delta \subseteq \Omega_1$, we have $Q \leq T$ and hence G/T is a p-group, because $G = O_p(G)Q$.

Choose $g \in G \setminus T$ such that Tg moves 1. Any cycle of Tg has length a power of p. Without loss of generality, let $(1, 2, \ldots, p^e)$ be such a cycle. Now $\mathrm{supp}_\Omega(Q^{g^i}) \subseteq \Omega_1 g^i$ for all integers $i \geq 0$. If $0 \leq i < j < p^e$ then $\Omega_1 g^i \cap \Omega_1 g^j = \emptyset$ since $\Omega_1 g^i$ and $\Omega_1 g^j$ are distinct blocks. Thus elements of Q^{g^i} commute with elements of Q^{g^j}. Hence $B = Q \times Q^g \times \ldots \times Q^{g^{p^e-1}}$ is a subgroup of G and g cyclically permutes the terms $Q, Q^g, \ldots, Q^{g^{p^e-1}}$. There is an epimorphism $\langle Q, g \rangle = \langle B, g \rangle \to Q \wr C_{p^e} \cong C_q \wr C_{p^e}$. By hypothesis, $\langle Q, g \rangle$

is supersoluble. This contradicts 4.1.1 because here $q < p$ and for $C_q \wr C_{p^e}$ to be supersoluble we need $p^e | q - 1$.

In either case we get a contradiction. Therefore $G = N = O_p(G)$ is a p-group. □

The finite situation is, of course, very different. For example, Sym(3) is a transitive supersoluble group that is not a p-group for any prime p.

Note that G and Ω in 4.1.2 are countably infinite by 1.1.3 Part 3.

4.1.3. Theorem. *Let V be an infinite-dimensional vector space over the division ring D and let G be a locally supersoluble irreducible subgroup of* FGL(V). *Then G is a locally finite p-group for some prime $p \neq$* char$D$.

Proof. Again we have $G' \leq \eta(G) = N$ and by Lemma 1.2.3, N is irreducible. By Wehrfritz's theorem (in [46]), N is a locally finite p-group for some prime $p \neq \mathrm{char} D$.

Let A/N be the p-primary component of the abelian group G/N. Then A/N is locally finite and so A is a locally finite normal p-subgroup of G. Therefore A is locally nilpotent and by the maximality of N, we have $N = A$. Thus $N = O_p(G)$. Once we have shown that G is a p-group, then the local finiteness follows. For G/N is then a periodic abelian group and thus is locally finite. Since also N is locally finite, G is locally finite.

Suppose for a contradiction that $N \neq G$. By 1.3.4, G is imprimitive. Choose any proper G-system of imprimitivity $(V_\omega)_{\omega \in \Omega}$ of V. Let $T = \bigcap_{\omega \in \Omega} N_G(V_\omega)$. Using 1.2.1, Ω is an infinite set and G/T is isomorphic to a transitive subgroup of FSym(Ω). By 4.1.2, G/T is an r-group for some prime r. Now $NT/T \cong N/(N \cap T)$ is both a p-group and an r-group.

Thus $p = r$ or $N = N \cap T$. However, if $N = N \cap T$ then $G' \leq N \leq T$ and then G/T is abelian, contradicting 1.1.5. Therefore G/T is a p-group.

Let x be an element of G of order q where $q = \infty$ or $p \neq q$ is a prime. Let $Q = \langle x \rangle$. Clearly $N \cap Q = 1$ so $N \rtimes Q = NQ \leq G$.

Case 1: $q = \infty$ and $p = 2$, or $q < \infty$ and $q \not| p - 1$. Let X be a finitely generated subgroup of G. Now we can find a subgroup Y of G containing X which is generated by x together with finitely many elements of N. The subgroup Y is supersoluble, so $Y \cap N$ has a Y-supersoluble series whose factors are C_p. Also x acts as an automorphism on these factors. In the case where q is infinite and $p = 2$, we note that $\mathrm{Aut} C_p$ is trivial. Thus x centralizes the Y-supersoluble series of $Y \cap N$, so that $Y = (Y \cap N) \rtimes \langle x \rangle$ is nilpotent. The case where q is finite and $q \not| p - 1$ is handled as in 4.1.2. Either way, we conclude that G is locally nilpotent, a contradiction.

Case 2: $q = \infty$ and $p \neq 2$, or $q < \infty$ and $q | p - 1$. Note that for these values, $C_q \wr C_{p^e}$ is never supersoluble. When q is finite, this is immediate from 4.1.1. The infinite case holds because there is an epimorphism $C_\infty \wr C_{p^e} \to C_2 \wr C_{p^e}$.

Now Q acts non-trivially on only finitely many of the V_ω, say $V_{\omega_1}, V_{\omega_2}, \ldots,$ V_{ω_m}. Put $\Delta = \{\omega_1, \ldots, \omega_m\}$. Since G/T must be totally imprimitive (by 1.1.2 and 1.1.3 Part 1), there is a G-congruence with blocks $(\Omega_i)_{i \in I}$ with $\Delta \subseteq \Omega_1$ by 1.1.3. Set $S = \bigcap_{i \in I} N_G(\Omega_i)$. Then by 1.1.1, G/S is a transitive subgroup of $\mathrm{FSym}(I)$.

Let $g \in G$ be such that Sg moves 1. Then Sg contains a cycle $(1, 2, \ldots, p^e)$ for some integer $e \geq 1$. Put $W_i = \bigoplus_{\omega \in \Omega_i} V_\omega$. Now Q^{g^i} centralizes W_j when $j \neq i$. Thus $[Q^{g^i}, Q^{g^j}] = 1$ for $0 \leq i < j < p^e$. As in the proof of 4.1.2, there is an epimorphism $\langle Q, g \rangle \to C_q \wr C_{p^e}$ and thus $\langle Q, g \rangle$ cannot be supersoluble, a contradiction. $\square$

The finite-dimensional skew linear situation is much different; for any locally supersoluble torsion-free group has a faithful irreducible skew linear representation of degree 1 by [34] 1.4.8 and there are such groups which are not locally nilpotent (for example $\langle x, y : x^y = x^{-1} \rangle = C_\infty \rtimes C_\infty$).

4.1.4. Corollary. *Let G be an irreducible locally supersoluble subgroup of $\mathrm{FGL}(V)$ where V has infinite-dimension over D. Then $\dim_D V$ is countable, G is a countably infinite locally finite p-group, G is isomorphic to a completely reducible monomial finitary linear group over any algebraically closed field of characteristic not p and G embeds into $\mathrm{GL}(m, D) \wr_\Omega S$ where Ω is a countably infinite set, m is some natural number and S is a transitive p-subgroup of $\mathrm{FSym}(\Omega)$.*

Proof. Since G is locally nilpotent by 4.1.3, all this follows from the theorem in [46]. $\square$

4.2 Some General Remarks

1. If one restricts D to being a field in 4.1.3 then there is a simpler proof. By 2.2.5, G has an abelian normal subgroup A such that G/A is a transitive finitary permutation group of infinite degree. Using 4.1.2, G/A is necessarily a q-group and using Wehrfritz's result ([46] Theorem), $\eta(G)$ is a p-group for primes q, p where $p \neq \mathrm{char}D$. Now $A \leq \eta(G)$ and since $\eta(G)$ is irreducible (it contains G'), $A \neq \eta(G)$ by 1.3.3. Thus $p = q$ as required.

2. A finitary p-group over D where $\mathrm{char}D = p > 0$ is a stability group. Now stability groups cannot act irreducibly unless they are trivial (see Theorem 1.2.6 Part 2), so the prime p cannot be $\mathrm{char} D$ in 4.1.3.

3. Such groups in 4.1.2 and 4.1.3 exist. Let $W = \mathrm{Wr}_{(\mathbb{N},<)} C_p$ be the wreath power of the cyclic group of prime order p over the natural numbers with their natural ordering. Then W is a transitive finitary permutation group on $\mathbb{N}$. Let H be any transitive finitary permutation p-group on a countably infinite set Ω and let F be an algebraically closed field of characteristic not p. Then $A = C_p$ is a subgroup of $F^* = \mathrm{GL}(1, F)$. Let $G = A \wr_\Omega H$. The base group of G acts componentwise on the space of row vectors $V = \bigoplus_{\omega \in \Omega} F_\omega$, where F_ω is an isomorphic copy of F, and H permutes the components F_ω in the obvious way. Moreover, this action is finitary and irreducible, i.e. G is an irreducible subgroup of $\mathrm{FGL}(V)$.

4. In [46], a crucial step in the proof that a locally nilpotent infinite-dimensional irreducible finitary skew linear group is a p-group, is to show that the group is periodic. This is done using the following result:

4.2.1. Lemma ([46] Lemma 4). *Let G be a locally nilpotent homogeneous subgroup of $\mathrm{FGL}(V)$ and let $g \in G$ be an element of infinite order. Then $[V, g] = V$ and $C_V(g) = 0$.*

This result does not extend to locally supersoluble groups, even when the group is linear and supersoluble. For let,

$$G = \left\langle g = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \cong C_\infty \wr C_2.$$

Then G is irreducible as a subgroup of $\mathrm{GL}(2, \mathbb{C})$, is supersoluble and g has infinite order. But if $V = \mathbb{C} \oplus \mathbb{C}$, then $[V, g] < V$ and $0 < C_V(g)$.

However, we have the following:

4.2.2. Proposition. *Let G be a non-trivial locally supersoluble homogeneous subgroup of* $\mathrm{GL}(n, D)$ *and let* $V = D^{(n)}$ *be the space of n-dimensional row vectors over D. Then there is $x \in G$ with $[V, x] = V$ and $C_V(x) = 0$.*

The proof is essentially that of [46] Lemma 5 but we include it here.

Proof. There is a finitely generated subgroup X of G which is homogeneous, by [41] 8b. Now X is supersoluble, so it has a non-trivial cyclic normal subgroup generated by x, say. Choose any D-X irreducible submodule W of V. Then X must act faithfully on W since X is homogeneous. Thus $[W, x]$ is non-trivial. Since $\langle x \rangle \lhd X$, it follows that $[W, x]$ is a D-X submodule of W and hence $W = [W, x]$. Using the homogeneity of X, we get $[V, x] = V$. The last part follows because $C_V(x)$ is the kernel of the map $V \to [V, x]$, $v \mapsto [v, x]$. □

5. The corollary in [46] extends partially:

4.2.3. Proposition. *Suppose D is a division ring of characteristic $p \geq 0$ that is locally finite-dimensional over a subfield. Let G be a completely reducible locally supersoluble subgroup of* $\mathrm{FGL}(V)$ *and let X be a finitely generated subgroup of G. Then X is completely reducible under any one of the following circumstances:*

1. *V is a direct sum of infinite-dimensional D-G irreducibles,*
2. *the characteristic p is 0, or*
3. *$p \neq 0$ and X has no normal subgroup of index p.*

Proof. We may assume that G is irreducible. If $\dim_D(V) = \infty$ then by 4.1.3 and the corollary in [46], X is completely reducible. Thus assume that V has finite D-dimension and set $N = \eta(G)$.

By 2.3.3, G/N is periodic abelian. Now $X/(X \cap N) \cong XN/N$ is finitely generated periodic abelian and thus is finite. By a famous theorem of Schreier, $X \cap N$ is a finitely generated subgroup of N. By Theorem 1.2.2, N is completely reducible and so by the corollary in [46] again, $X \cap N$ is completely reducible. The result now follows using the proof of Maschke's theorem (see [34] 1.1.3). $\square$

The restrictions 1-3 in the statement of 4.2.3 are necessary. Let p be a prime. By a theorem of Dirichlet (see for example [12] Theorem 15) there is a prime q for which p divides $q-1$. Choose F to be any field of characteristic p containing a primitive q-th root of unity α and let $h = (1, 2, ..., p)$. Put $A = \langle \alpha \rangle \leq F^*$, $H = \langle h \rangle$ and $G = A \wr H$. Then G may be regarded as an irreducible subgroup of $\mathrm{GL}(p, F)$. Using 4.1.1, G is supersoluble. Now in $\mathrm{End}_F(V)$, $(h^i - 1)^p = h^{ip} - 1 = 0$ for any integer i, so H is unipotent and therefore is not completely reducible.

If K is any irreducible locally nilpotent subgroup of $\mathrm{FGL}(W)$ where W has infinite dimension over F, then $G \times K$ is a completely reducible locally supersoluble subgroup of $\mathrm{FGL}(V \oplus W)$ which has a finitely generated subgroup that is not completely reducible.

4.3 Appendix

The results of this chapter were submitted as a paper to Archiv der Mathematik. The anonymous referee made some suggestions which I used in the final edition. The following work appears in [25].

4.3.1. Theorem. *Let G be a locally-nilpotent by abelian group.*

1. *If G is a transitive finitary permutation group of infinite degree then G is a p-group for some prime p.*

2. *If G is an irreducible finitary skew linear group of infinite dimension over the division ring D, then G is a locally finite p-group for some prime $p \neq \mathrm{char} D$.*

Clearly Theorems 4.1.2 and 4.1.3 follow from 4.3.1.

4.3.2. Corollary. *Let G be either a transitive finitary permutation group of infinite degree or an irreducible finitary skew linear group of infinite dimension over a division ring D which is locally finite-dimensional over a subfield. If $G/\eta(G)$ satisfies a non-trivial law, then G is a locally finite p-group for some prime $p \neq \mathrm{char} D$.*

We shall prove Corollary 4.3.2 first. In either case $\eta(G) \neq 1$, for otherwise G satisfies a non-trivial law and this is not possible by [20] Theorem 1 and Theorem 3.4.2. In the finitary permutation case, by 1.1.6 $\eta(G)$ is transitive and so by 1.1.7, we have $G' \leq \eta(G)$. In the finitary skew linear case, $\eta(G)$ is irreducible using 1.2.3 and by 1.3.4 and 1.2.4, $G' \leq \eta(G)$. Therefore G is locally-nilpotent by abelian in either case. To finish we apply 4.3.1.

We now head towards a proof of 4.3.1.

4.3.3. Lemma. *Let G be a transitive subgroup of* $\mathrm{FSym}(\Omega)$ *where* $|\Omega| = \infty$ *and suppose that G' is a p-group. Then G is a p-group.*

Proof. Let $g \in G$ and put $\Gamma = \mathrm{supp}_\Omega(g)$. By Neumann's Lemma 1.1.4, there is $x \in G$ with $\Gamma \cap \Gamma x = \emptyset$. Now $\mathrm{supp}_\Omega(g^{-1}) = \Gamma$ and $\mathrm{supp}_\Omega(g^x) = \Gamma x$, so that g^{-1} and g^x commute. Thus the commutator $[g, x] = g^{-1}g^x$ has the same order as that of g. Since G' is a p-group, it follows that g has order a power of p, as required. □

4.3.4. Lemma. *Let G be an irreducible subgroup of* $\mathrm{FGL}(V)$ *where* $\dim_D V = \infty$ *and suppose that G' is a locally finite p-group. Then G is a locally finite p-group.*

Proof. By Theorem 1.3.4, G is imprimitive. Choose a proper G-system of imprimitivity $(V_\omega)_{\omega\in\Omega}$ of V. Necessarily Ω is infinite and the action of G on this system induces a transitive finitary action on Ω. Let $g \in G$. Then $\Gamma = \{\omega \in \Omega | g \notin C_G(V_\omega)\}$ is a finite set. By 1.1.4, there is $x \in G$ with $\Gamma \cap \Gamma x = \emptyset$. Now g^{-1} centralizes all the V_ω with $\omega \notin \Gamma$ and g^x centralizes all the V_ω with $\omega \notin \Gamma x$. Thus, as in 4.3.3, $[g, x]$ has the same order as g and it follows that G is a p-group. Since G' is locally finite and G/G' is periodic abelian, G is locally finite. □

Proof of the Theorem. Now G' is locally nilpotent. In the finitary permutation case, G' is transitive by 1.1.6 and in the finitary skew linear case, G' is irreducible by 1.2.3. Thus by either Suprunenko's Theorem or Wehrfritz's Theorem, G' is a locally finite p-group. The Theorem follows from 4.3.3 and 4.3.4. □

Also the following is of interest.

4.3.5. Remark. *Let G be a locally finite p-group.*

1. *If G is a transitive subgroup of $W = \mathrm{FSym}(\Omega)$ where Ω is infinite, then $N_W(G)$ is a p-group for some prime p.*

2. *If G is an irreducible subgroup of $W = \mathrm{FGL}(V)$ where V has infinite dimension, then $N_W(G)$ is a locally finite p-group for some prime p.*

Proof. Let $x \in N_W(G)$ and $H = \langle G, x \rangle$. Certainly $H' \leq G \lhd H$. Now apply Lemmas 4.3.3 and 4.3.4 to H in each case. □

Paul Flavell has pointed out to me an easier way to finish the proofs of 4.1.2 and 4.1.3. In 4.1.2, case 2 can be finished by using the fact that $B \cap O_p(G) = 1$. This forces B to be cyclic and it clearly isn't. A similar argument can be used in 4.1.3.

Chapter 5

Height

5.1 Paraheight

In [38], Wehrfritz defines a supersoluble analogue of central height called *paraheight*. Let G be a group and let A be a G-module. Then A is said to be *G-complete* if every subgroup of A (as an abelian group) is a G-submodule of A. Abelian normal sections of G are G-modules via conjugation. Clearly, an abelian normal subgroup A of G is G-complete if and only if all its subgroups are normal in G. Thus, completeness in G can be thought of as a generalization of centrality in G.

Let $N \lhd G$. A *G-paraseries of N of height β* is an ascending series of N

$$1 = N_0 \leq N_1 \leq \ldots \leq N_\alpha \leq \ldots \leq N_\beta = N$$

such that each $N_\alpha \lhd G$, each $N_{\alpha+1}/N_\alpha$ is abelian and G-complete. If N possesses a G-paraseries then the *G-paraheight* of N is the smallest ordinal β for which a G-paraseries of N of height β exists. Note that a G-paraseries is sometimes called a "parasoluble series" (cf. [13]).

Using the fact that every subgroup of a cyclic group is characteristic, it is easy to see that any G-hypercyclic series is a G-paraseries. In particular, the subgroup $\lambda(G)$ always has a G-paraseries. The G-paraheight of $\lambda(G)$ is called the *paraheight* of G.

Wehrfritz introduced paraheight in order to generalize the work of Gruenberg on centrality in linear groups (see [9]). Gruenberg proved that the central height of a linear group is strictly less than $\omega 2$. Wehrfritz proved the following for paraheight:

5.1.1. Theorem (Wehrfritz, [38]). *Let G be a linear group of degree n. Then G has paraheight at most $\omega + \lfloor \log_2 n! \rfloor$.*

It is still unknown as to whether 5.1.1 gives the correct bound. Banu ([1] page 64) has shown that if G is a hypercyclic linear group of degree n then G has paraheight $\leq \omega + \lfloor \frac{1}{2}(n+1) \rfloor$ and that this bound is attained when n is a power of 2 bigger than 4. Stewart, in his PhD thesis [35], examined paraheight in skew linear groups and found that there is no bound on paraheights of skew linear groups.

A natural thing to investigate is what happens in the finitary linear case. Gruenberg's theorem on linear groups extends to finitary linear groups provided one ignores unipotence:

5.1.2. Theorem (Wehrfritz, [45]). *A finitary linear group has central height less than or equal to $\omega 2$ modulo its unipotent radical.*

One technique in computing paraheights of hypercyclic groups is the following:

5.1.3. Lemma. *Let G be a hypercyclic group and let α be an ordinal. Let p be the paraheight of G. If $p = \alpha + 1$ then G' has G-paraheight α. If G' has G-paraheight α then $\alpha \leq p \leq \alpha + 1$, and $p = \alpha + 1$ if α is not a limit ordinal or zero.*

The group $G = C_{3^\infty} \wr C_3$ has paraheight $\omega + 1$ and G' has G-paraheight ω (see [39] proof of 11.25 on page 171). We shall see that the direct power P of $\aleph_0$ copies of Sym(3) has paraheight ω and its commutator has P-paraheight ω also. Thus 5.1.3 is the best we can do.

In part of the above lemma, it is an implicit assumption that G is imperfect. If G is a perfect hypercyclic group then G is hypercentral. In this case, it follows from Grün's Lemma (see [30] Part I page 48) that G is trivial.

Proof. If G' has G-paraheight α, then G certainly has paraheight $\leq \alpha + 1$.

Suppose that

$$1 = G_0 < G_1 < G_2 < \ldots < G_\alpha < G_{\alpha+1} = G$$

is a G-paraseries of G. Then $G' \leq G_\alpha$. Consider the series

$$1 = G_0 \cap G' \leq G_1 \cap G' \leq \ldots \leq G_\alpha \cap G' = G'.$$

The terms of this series are normal in G and the series has abelian factors. Let $G_\beta \cap G' \leq H \leq G_{\beta+1} \cap G'$ for an ordinal $\beta < \alpha$. Now $G_\beta \leq G_\beta H \leq G_{\beta+1}$, so $G_\beta H \lhd G$. Also

$$H = (G' \cap G_\beta)H = G' \cap HK_\beta \lhd G.$$

This argument shows that the factors $\frac{G_{\beta+1} \cap G'}{G_\beta \cap G'}$ are G-complete and thus G' has G-paraheight $\leq \alpha$.

Now if G' has G-paraheight $< \alpha$ then G has paraheight $< \alpha+1$. Therefore G has paraheight $\alpha + 1$ implies that G' has G-paraheight exactly α. If α is not a limit ordinal or zero, then G cannot have paraheight α; otherwise, G' has G-paraheight $\alpha - 1$. $\square$

Finitary linear groups can contain very large direct products and paraheight does not respect direct products.

5.1.4. Example. *Let δ be any cardinal and let G_δ be the direct product of δ copies of* Sym(3). *Then the paraheight of G_δ is $\delta + 1$ if δ is finite, and is δ if δ is infinite. Also G_δ is a finitary linear group with $\mathcal{U}(G_\delta) = 1$. In particular, $G_{\aleph_1}$ is a finitary linear group of paraheight $> \omega 2$.*

Proof. We show that $P = (G_\delta)'$ has G_δ-paraheight δ. Now in this case, P is the direct product of cyclic groups of order 3. Let C be a G-complete normal subgroup of P. If $x = (x_\epsilon)_{\epsilon \leq \delta} \in C \setminus 1$ with $x_{\epsilon_1} \neq 1 \neq x_{\epsilon_2}$ for some $\epsilon_1 < \epsilon_2$ and $g \in G$ is an involution in the ϵ_1th copy of Sym(3), then

$$x^g = (\ldots, x_{\epsilon_1}^{-1}, \ldots, x_{\epsilon_2}, \ldots) \notin \langle x \rangle .$$

Thus C must be one of the cyclic direct factors of P.

Given a G-paraseries

$$1 = N_0 \leq N_1 \leq \ldots \leq N_\alpha$$

contained in P for ordinal $\alpha \geq 1$, we want to produce another term $N_{\alpha+1} \leq P$ to extend the G-paraseries. By induction N_α is a product of direct factors of P with each $N_{\beta+1}/N_\beta$ cyclic of order 3 for $\beta < \alpha$. Consider G/N_α. By considering the action of G/N_α on P/N_α, the only candidate for $N_{\alpha+1}$ is a direct product of one more of the direct factors of P with N_α. Hence $N_{\alpha+1}/N_\alpha$ is cyclic of order 3.

For limit ordinals α, the only possibility for a term in an ascending series is the union

$$\bigcup_{\beta<\alpha} N_\beta$$

which is clearly a direct product of direct factors of P.

This induction stops with $N_\delta = P$. Thus P has G-paraheight exactly δ. By 5.1.3, G_δ has paraheight $\delta+1$ for finite δ.

In the case where δ is infinite, G_δ has paraheight at least δ. Also by considering the following G_δ-paraseries we see that G_δ has paraheight exactly δ:

$$1 \le C_3 \times 1 \times 1 \times \ldots \le \mathrm{Sym}(3) \times C_3 \times 1 \times \ldots \le \bigtimes_{\aleph_0} \mathrm{Sym}(3) \le$$

$$\bigtimes_{\aleph_0} \mathrm{Sym}(3) \times C_3 \le \bigtimes_{\aleph_0} \mathrm{Sym}(3) \times \mathrm{Sym}(3) \times C_3 \le \ldots \le G_\delta.$$

That G_δ is a unipotent-free finitary linear group is clear; Sym(3) is a unipotent-free linear group over $\mathbb{C}$ and so G is finitary linear and unipotent-free over $\mathbb{C}$ in the natural way. □

In contrast to Example 5.1.4, any direct power of D_8 has central height and paraheight 2. One feels that any measure of supersolubility should also be 2 in the case of Example 5.1.4.

There is no general canonical G-paraseries of a group G which contains all G-paraseries. For example, if

$$G = Q_8 = \left\langle x, y : x^4 = y^4 = 1, x^2 = y^2, xy = (yx)^{-1} \right\rangle$$

then both $1 < \langle x \rangle < G$ and $1 < \langle y \rangle < G$ are G-paraseries. The only series of G containing both of these is $1 < G$ and this certainly isn't a G-paraseries. Hill defines the notion of an "upper paraseries" (see [13] page 44 where this is called an "upper parasoluble series"). Both series $1 < \langle x \rangle < G$ and $1 < \langle y \rangle < G$ are upper paraseries in his sense; it is the lack of uniqueness of these upper paraseries that is the problem.

It is natural to ask whether there is a type of series which measures the supersoluble height of groups and such that every group has a canonical supremum for these series (like the upper central series being a canonical supremum for all central series).

In this chapter, we shall present two alternatives to paraheight. The first alternative works with respect to direct products and works for finitary linear groups (again, provided one ignores unipotence). It does seem slightly artificial. The second alternative gives a generalization of the upper central series.

5.2 Weak paraheight

In this section, we weaken the definition of paraheight in an attempt to get a reasonable result for finitary linear groups.

Let $N \lhd G$. A *G-weak-paraseries of N of height β* is an ascending series

$$1 = N_0 \leq N_1 \leq \ldots \leq N_\alpha \leq \ldots \leq N_\beta = N$$

of normal subgroups N_α of G, such that each factor $N_{\alpha+1}/N_\alpha$ is abelian and, as a G-module, embeds into a direct sum of G-complete modules.

Suppose that we have such a series

$$1 = N_0 \leq N_1 \leq \ldots \leq N_\alpha \leq \ldots N_\beta = N$$

and $N_{\alpha+1}/N_\alpha \hookrightarrow \bigoplus_{j \in I} M_j$ via the G-map φ where each M_j is a G-complete module. Let π_i be the natural projection $\bigoplus_{j \in I} M_j \to M_i$. We can choose M_i to be the images of $M = N_{\alpha+1}/N_\alpha$ under $\varphi\pi_i$. Now $M\varphi\pi_i \leq M_i$ and furthermore $M\varphi\pi_i$ is G-complete. The map

$$\psi : M \to \bigoplus_{i \in I} M\varphi\pi_i, \;\; m \mapsto (m\varphi\pi_i)_{i \in I}$$

is a G-module homomorphism with kernel

$$\begin{aligned} \ker\psi &= \{x \in M : x\varphi\pi_i = 1, \text{ for all } i \in I\} \\ &= \{x \in M : x\varphi \in \cap_{i \in I} \ker\pi_i = 1\} \\ &= \ker\varphi. \end{aligned}$$

Thus M embeds into $\bigoplus_{i \in I} M\varphi\pi_i$.

The least ordinal β for which a G-weak-paraseries of N of height β exists, is called the *G-weak-paraheight* of N.

Trivially, a G-paraseries is a G-weak-paraseries. Hence any G-hypercyclic normal subgroup of G has a G-weak-paraseries. In fact, the converse is true; any direct sum of G-complete modules is G-hypercyclic, so any factor of a G-weak-paraseries is G-hypercyclic.

Now $\lambda(G)$ has a G-weak-paraseries. The G-weak-paraheight of $\lambda(G)$ is called the *weak paraheight* of G. Clearly, the weak paraheight of G is less than or equal to the paraheight of G.

It is easy to see that in the case of Example 5.1.4, each group G_δ has weak paraheight 2; in that case $(G_\delta)'$ is a direct product of G_δ-complete modules.

We list a few basic properties of weak paraseries.

5.2.1. Lemma. *Let $N \lhd G$ have a G-weak-paraseries of length β. Suppose that $K \lhd G$ and $H \leq G$. Then:*

1. *$K \cap N$ has a G-weak-paraseries of length β;*

2. *$H \cap N$ has an H-weak-paraseries of length β;*

Proof. Let

$$1 = N_0 \leq N_1 \leq \ldots \leq N_\alpha \leq \ldots \leq N_\beta = N$$

be a G-weak-paraseries of length β.

1. Let $K_\alpha = K \cap N_\alpha, \lhd G$. Then

$$\frac{K_{\alpha+1}}{K_\alpha} \cong_G \frac{(N_{\alpha+1} \cap K)N_\alpha}{N_\alpha}.$$

Thus $K_{\alpha+1}/K_\alpha$ is abelian and embeds into a direct sum of G-complete modules.

2. The proof of 2 is similar to that of 1.

□

5.2.2. Lemma. *Let $(G_i)_{i\in I}$ be a family of groups of weak paraheight at most β. Then their direct product G has weak paraheight at most β.*

Proof. For each $i \in I$, let

$$1 = N_0^{[i]} \leq N_1^{[i]} \leq \ldots \leq N_\alpha^{[i]} \leq \ldots \leq N_\beta^{[i]} = \lambda(G_i)$$

be a G_i-weak-paraseries for $\lambda(G_i)$. These are G-weak-paraseries.

Each factor of the series

$$1 = \bigtimes_{i\in I} N_0^{[i]} \leq \bigtimes_{i\in I} N_1^{[i]} \leq \ldots \leq \bigtimes_{i\in I} N_\alpha^{[i]} \leq \ldots \leq \bigtimes_{i\in I} N_\beta^{[i]} = \bigtimes_{i\in I} \lambda(G_i)$$

is certainly abelian.

We also have the following equality:

$$\bigtimes_{i\in I} \lambda(G_i) = \lambda(G).$$

(Certainly, each $\lambda(G_i)$ is G-hypercyclic. To see the converse, consider the projections of $\lambda(G)$ into the G_i.)

Now each $N_{\alpha+1}^{[i]}/N_\alpha^{[i]}$ embeds into a direct sum of G-complete modules, so $\bigtimes_{i\in I} N_{\alpha+1}^{[i]}/N_\alpha^{[i]}$ embeds into a direct sum of G-complete modules. The result follows. □

Using these properties, we can prove the following:

5.2.3. Theorem. *Let G be a finitary linear group. Then $G/\mathcal{U}(G)$ has weak paraheight at most $\omega 2$.*

Proof. We may assume that $\mathcal{U}(G) = 1$ and further that G is completely reducible. Decompose V into a direct sum

$$V = \bigoplus_{i\in I} V_i$$

of irreducible FG-submodules V_i. Now consider the homomorphism

$$\varphi : G \to \prod_{i\in I} \frac{G}{C_G(V_i)};\ g \mapsto (C_G(V_i)g)_{i\in I}.$$

Using the finitariness of G, the image $\overline{G}$ of G under φ must lie inside $P = \bigtimes_{i\in I} \frac{G}{C_G(V_i)}$; that is to say each $g \in G$ lies in all but finitely many of the $C_G(V_i)$. Also the kernel of φ is $\bigcap_{i\in I} C_G(V_i) = 1$.

Now $\lambda(\overline{G})$ is the image of $\lambda(G)$ under φ. Also $x \in \lambda(G)$ if and only if $(C_G(V_i)x)_{i\in I} \in \lambda(\overline{G})$. Let $x \in \lambda(G)$. Then for all $i \in I$, we have $C_G(V_i)x \in \lambda(G/C_G(V_i))$. Thus $\lambda(\overline{G}) \leq \lambda(P)$.

The subgroup $\lambda(P) \cap \overline{G}$ is a $\overline{G}$-hypercyclic subgroup of $\overline{G}$ so it follows that

$$\lambda(P) \cap \overline{G} = \lambda(\overline{G}).$$

Hence by 5.2.1 Part 2, if P has weak paraheight $\leq \omega 2$ then $\overline{G}$, and thus G, has weak paraheight $\leq \omega 2$. So it remains to show that P has weak paraheight $\leq \omega 2$.

Let H be an irreducible finitary linear group. Now if H has finite dimension, then by Wehrfritz's result 5.1.1, H has (weak) paraheight $< \omega 2$. If H has infinite dimension, it contains no cyclic normal subgroups (see 2.2.1), so has zero (weak) paraheight.

Now each $G/C_G(V_i)$ acts faithfully, linearly and irreducibly on V_i. By 5.2.2, P has weak paraheight at most $\omega 2$. The result follows.

□

We conclude this section with some open questions.

1. Is $\omega + \lfloor \frac{1}{2}(n+1) \rfloor$ the correct bound for weak paraheight of hypercyclic linear groups of degree n?

2. Let D be a locally finite-dimensional division algebra. Is the weak paraheight of a skew linear group over D of degree n bounded in terms of n? If so, the proof of Theorem 5.2.3 should work immediately to obtain a bound for unipotent-free finitary skew linear groups over D.

5.3 The Lambda series

In this section, we look at a different type of series that measures supersolubility within groups.

Let G be a group. We define a sequence of subgroups $\lambda_\alpha(G)$ as follows. Let $\lambda_0(G) = 1$, and given $\lambda_\alpha(G)$ for any ordinal α, define $\lambda_{\alpha+1}(G)$ to be such that $\lambda_{\alpha+1}(G)/\lambda_\alpha(G)$ is the subgroup generated by all cyclic normal subgroups of $G/\lambda_\alpha(G)$. At limit ordinals β, we put $\lambda_\beta(G) = \bigcup_{\alpha<\beta} \lambda_\alpha(G)$. In particular, $\lambda_1(G)$ is generated by the cyclic normal subgroups of G.

Each subgroup $\lambda_\alpha(G)$ is characteristic in G and for each ordinal α,

$$\lambda_{\alpha+1}(G)/\lambda_\alpha(G) = \lambda_1(G/\lambda_\alpha(G)).$$

Also, each $\lambda_\alpha(G)$ is G-hypercyclic and as such is contained in $\lambda(G)$.

Now for some ordinal β_0, we have $\lambda_{\beta_0}(G) = \lambda_{\beta_0+1}(G)$; for example, take β_0 to be any cardinal greater than $|G|$. Thus there is a least ordinal β for which $\lambda_\beta(G) = \lambda_{\beta+1}(G)$, which we call the *cyclic height* of G. It follows that $\lambda(G) = \lambda_\beta(G)$; if not, we can choose a non-trivial cyclic normal subgroup of $G/\lambda_\beta(G)$ and hence $\lambda_{\beta+1}(G) > \lambda_\beta(G)$.

The series

$$1 = \lambda_0(G) \leq \lambda_1(G) \leq \ldots \leq \lambda_\alpha(G) \leq \ldots \leq \lambda_\beta(G) = \lambda(G)$$

is the supremum for series whose factors are generated by certain cyclic normal subgroups. We make this precise.

5.3.1. Proposition. *Let $N \lhd G$ and suppose we have an ascending series*

$$1 = N_0 \leq N_1 \leq \ldots \leq N_\alpha \leq \ldots \leq N_\beta = N$$

where each $N_\alpha \lhd G$ and each $N_{\alpha+1}/N_\alpha$ is generated by some (but not necessarily all) cyclic normal subgroups of G/N_α. For every ordinal α, we have

$$N_\alpha \leq \lambda_\alpha(G).$$

Proof. We prove this result using transfinite induction. Clearly if $\alpha = 0$ then the result is true. If γ is a limit ordinal and $N_\alpha \leq \lambda_\alpha(G)$ for every $\alpha < \gamma$ then $N_\gamma = \bigcup_{\alpha<\gamma} N_\alpha \leq \bigcup_{\alpha<\gamma} \lambda_\alpha(G) = \lambda_\gamma(G)$.

Now suppose that for α, $N_\alpha \leq \lambda_\alpha(G)$. Pick $x \in N_{\alpha+1}$ for which $\langle N_\alpha x\rangle \lhd G/N_\alpha$. Then $\langle N_\alpha, x\rangle \lhd G$, so that $\langle x\rangle \lambda_\alpha(G) = \langle N_\alpha, x\rangle \lambda_\alpha(G) \lhd G$. Consequently $\langle \lambda_\alpha(G)x\rangle \lhd G/\lambda_\alpha(G)$. Therefore $x \in \lambda_{\alpha+1}(G)$. It follows that $N_{\alpha+1} \leq \lambda_{\alpha+1}(G)$. □

We shall call the following series the *Lambda series* of G.

$$\lambda_0(G) \leq \lambda_1(G) \leq \ldots \leq \lambda(G).$$

In [50] Chapter 1, Section 7, the above series is called the *ascending weakly central series* of G. Series of the type in 5.3.1 are called *weakly central* and 5.3.1 appears as Theorem 7.12.

If N is a G-complete normal subgroup of G, then N is generated by cyclic normal subgroups of G. Thus we have the following:

5.3.2. Corollary. *If*

$$1 = N_0 \leq N_1 \leq \ldots \leq N_\alpha \leq \ldots \leq N_\beta$$

is a G-paraseries, then $N_\alpha \leq \lambda_\alpha(G)$ for each ordinal α. In particular, the cyclic height of G is less than or equal to the paraheight of G. Also,

$$\zeta_\alpha(G) \leq \lambda_\alpha(G)$$

for every ordinal α. Thus, if G is hypercentral of central height α, then G is hypercyclic of cyclic height $\leq \alpha$.

So, at first glance, the Lambda series of a group seems to play a rôle similar to that of the upper central series. Before discussing how the Lambda series of a group relates to the Lambda series of its subgroups, quotients and similar derivatives, we list some properties of the series itself.

5.3.3. Proposition. *The series*

$$1 = \lambda_0(G) \leq \lambda_1(G) \leq \ldots \leq \lambda(G)$$

is G'-hypercentral. Furthermore, for each ordinal α,

$$\lambda_\alpha(G) \cap G' \leq \zeta_\alpha(G').$$

Proof. It is enough to prove that $\lambda_1(G)$ is G'-central. Let x be a element of G for which $\langle x \rangle \lhd G$. Now $\mathrm{Aut}(\langle x \rangle)$ is an abelian group and so if $y, z \in G$ then $x^{yz} = x^{zy}$. In other words, each commutator $[y, z]$ centralizes x and hence $[G', \lambda_1(G)] = 1$. □

Thus the 2nd term in the lower central series of $\lambda_1(G)$ is trivial since

$$\gamma^3(\lambda_1(G)) = [\lambda_1(G)', \lambda_1(G)] \leq [G', \lambda_1(G)] = 1.$$

In other words:

5.3.4. Corollary ([50] Chapter 1, Theorem 7.11). *Let G be any group. Then $\lambda_1(G)$ is nilpotent of class at most 2.*

The subgroup $\lambda_1(G)$ need not be abelian. For example, take $G = Q_8$. Every subgroup of G is normal, so consequently $\lambda_1(G) = G$. Thus the bound in

5.3.4 is the best possible. This is rather unfortunate; in studying solubility and related conditions, we would really like our building blocks to be the abelian groups. However, all is not lost:

5.3.5. Lemma. *Let G be a group. The derived subgroup $(\lambda_1(G))'$ is abelian and is generated by cyclic normal subgroups of G.*

Proof. This follows at once since

$$(\lambda_1(G))' = \langle [x,y] : \langle x \rangle , \langle y \rangle \lhd G \rangle ,$$

and since $\lambda_1(G)$ is nilpotent of class ≤ 2. □

In particular, $\lambda(G)$ has an ascending G-series

$$1 = N_0 \leq N_1 \leq \ldots \leq N_\alpha \leq \ldots \leq N_\beta = \lambda(G)$$

such that each factor $N_{\alpha+1}/N_\alpha$ is abelian and generated by some of the cyclic normal subgroups of G/N_α. If β is the smallest ordinal for which one of these series exists, we say that β is the *abelianized cyclic height* of G.

In general, there is no unique supremum for these abelianized series within a given group. The group Q_8 gives an example of this, as it did for paraseries.

The λ_1 operator respects direct products, like the ζ_1 operator.

5.3.6. Proposition. *Let $(G_i)_{i\in I}$ be a family of groups. Then*

$$\lambda_1(\bigtimes_{i\in I} G_i) = \bigtimes_{i\in I} \lambda_1(G_i).$$

Proof. Let $x \in G_i$ be such that $\langle x \rangle \lhd G_i$. Then $\langle x \rangle \lhd G$ and $x \in \lambda_1(G)$. Thus each $\lambda_1(G_i) \leq \lambda_1(G)$. Now pick $y \in G$ with $\langle y \rangle \lhd G$. Let $\pi_i : G \to G_i$ be the natural projection. Since $y = (y\pi_i)_{i\in I}$, we have $y \in \bigtimes_{i\in I} \lambda_1(G_i)$. □

Using a similar method to the above, one can show that

$$\lambda_1(\prod_{i\in I} G_i) \leq \prod_{i\in I} \lambda_1(G_i).$$

However the reverse inclusion does not necessarily hold - to see this, let $G = \prod_{\aleph_0} \mathrm{Sym}(3)$. Now any cyclic normal subgroup of G must be the cyclic group of order 3 in one of the direct factors $\mathrm{Sym}(3)$ of G. It follows that $\lambda_1(G) = \bigtimes_{\aleph_0} \lambda_1(\mathrm{Sym}(3)) \neq \prod_{\aleph_0} \lambda_1(\mathrm{Sym}(3))$.

The Lambda series behaves well with regard to images.

5.3.7. Lemma. *Let G be a group and let $N \lhd G$. Then for every ordinal α,*

$$\lambda_\alpha(G)N/N \leq \lambda_\alpha(G/N).$$

Proof. Given an ordinal α, consider $\lambda_{\alpha+1}(G)N/N$. We have

$$\frac{\lambda_{\alpha+1}(G)N/N}{\lambda_\alpha(G)N/N} = \langle (\lambda_\alpha(G)N/N)Nx : \langle x\rangle \lhd G\rangle .$$

Thus we can apply 5.3.1 to the series

$$1 = \frac{\lambda_0(G)N}{N} \leq \frac{\lambda_1(G)N}{N} \leq \ldots \leq \frac{\lambda_\alpha(G)N}{N} \leq \ldots \leq \frac{\lambda(G)N}{N}$$

and get $\lambda_\alpha(G)N/N \leq \lambda_\alpha(G/N)$ for every ordinal α. □

The Lambda series does not behave as well with respect to subgroups.

5.3.8. Example. *There is a finite supersoluble group G with a normal subgroup $H \lhd G$ such that G has cyclic height 1 but H has cyclic height 2.*

Proof. Let G be the following group:

$$G = \left\langle x, y, z | x^4 = y^4 = z^4 = 1, x^2 = y^2 = z^2, x^y = x^{-1}, x^z = x, y^z = y \right\rangle .$$

Premultiplying $xy = yx^{-1}$ by x^2 gives $x^{-1}y = x^2yx^{-1} = y^2yx^{-1} = y^{-1}x^{-1}$. Thus $y^x = y^{-1}x^{-1}x = y^{-1}$. Thus in G, x inverts y. Since z centralizes y, $\langle y \rangle \lhd G$.

From the relations, z centralizes x and y inverts it so that $\langle x \rangle \lhd G$. Also x and y centralize z so that $\langle z \rangle \lhd G$. Thus $\lambda_1(G) = G$.

Let $H = \langle x, h = yz \rangle$. Clearly H is a normal subgroup of G. Also H is dihedral of order 8. (For x has order 4, h has order 2 and $x^h = x^{yz} = x^{-z} = x^{-1}$.) A non-abelian dihedral group has cyclic height 2. □

However this example shows that the bound in the next result is the correct one.

5.3.9. Proposition. *Let G be any group and H be a normal subgroup of G. Then $\lambda_1(G) \cap H \leq \lambda_2(H)$.*

Proof. If $\langle x \rangle \lhd G$ then $[x, H] \leq \langle x \rangle \cap H$. Thus $[x, H]$ is cyclic and normal in H so that $[\lambda_1(G), H] \leq \lambda_1(H)$.

Now $[\lambda_1(G) \cap H, H] \leq [\lambda_1(G), H]$, so that $(\lambda_1(G) \cap H)/[\lambda_1(G), H]$ is central in $H/[\lambda_1(G), H]$ and, as such, is generated by cyclic normal subgroups of $H/[\lambda_1(G), H]$. Hence $\lambda_1(G) \cap H \leq \lambda_2(H)$ by 5.3.1. □

We would like to be able to prove that in a unipotent-free finitary linear group G, we have $\lambda(G) = \lambda_{\omega 2}(G)$, or at least something similar. Of course, this result holds for linear groups (without the unipotence restriction) by Wehrfritz's result on paraheight. In order to do the general case, it seems that one has to be able to relate the cyclic height of G to that of subgroups of G. If one could prove that if $H \leq G$, if β is a limit ordinal or is 0, and n

is a natural number, there is a natural number m such that

$$\lambda_{\beta+n}(G) \cap H \leq \lambda_{\beta+m}(H)$$

then this would be enough to obtain a proof of the above. We leave this problem as two open questions.

3. Is it possible to relate the Lambda series of G to the Lambda series of subgroups of G?

4. Let G be a finitary linear group (or a finitary skew linear group over a locally finite-dimensional division algebra) with $\mathcal{U}(G) = 1$. Is it true that the cyclic height of G is at most $\omega 2$? If not, is there a bound?

We conclude this chapter by looking at the Lambda series in linear groups.

5.3.10. Proposition. *There is no hypercyclic linear group of cyclic height exactly ω.*

Proof. Let G be a linear group such that $G = \lambda_\omega(G) = \bigcup_{i \in \mathbb{N}} \lambda_i(G)$. We shall show that G has finite cyclic height. By [39] Corollary 11.4, the unipotent radical of G has finite G-paraheight and so by Corollary 5.3.2 it is contained in $\lambda_n(G)$ for some natural number n. By 1.2.6, we may suppose that G is completely reducible.

Now by [39] Theorem 1.14, G is abelian-by-finite (it is monomial over the algebraic closure of the ground field and monomial linear groups are abelian-by-finite). Thus $G = \langle A, x_1, \ldots, x_m \rangle$ for some abelian normal subgroup A and some $x_1, \ldots, x_m \in G$. But the x_i all lie in finite-indexed terms of the Lambda series. There is a natural number k such that $\lambda_k(G)$ contains all the x_i. Hence $G = A\lambda_k(G)$, so that $G/\lambda_k(G)$ is abelian and $G = \lambda_{k+1}(G)$. □

However, there are linear groups with cyclic height exactly ω. Before investigating this, we note the following examples.

5.3.11. Example. *The group*

$$G = \left\langle (x,y), g : x, y \in C_{3^\infty}, g^3 = 1, (x,y)^g = (y^{-1}, y^{-1}x) \right\rangle = (C_{3^\infty} \times C_{3^\infty}) \rtimes C_3$$

has (abelianized) cyclic height exactly $\omega + 1$ *and* $\lambda_\omega(G)$ *is*

$$B = \{(x,y) : x, y \in C_{3^\infty}\}.$$

Proof. We show first that every cyclic normal subgroup of G lies in B. Suppose that $x, y \in C_{3^\infty}$. If $\langle g^2(x,y)\rangle \lhd G$ then $\langle g(y, x^{-1}y)\rangle \lhd G$, since $(g^2(x,y))^2 = g(y, x^{-1}y)$. Because the order of g is 3, it is enough to consider elements of the form $g(x,y)$.

Note that

$$\begin{aligned} (g(x,y)^3) &= g^3(x,y)^{g^2}(x,y)^g(x,y) \\ &= (x^{-1}y, x^{-1}yy^{-1})(y^{-1}, y^{-1}x)(x,y) \\ &= 1. \end{aligned}$$

So the only non-trivial powers of $g(x,y)$ are $g(x,y)$ itself and $(g(x,y))^2 = g^2(y^{-1}x, x)$. If $\langle g(x,y)\rangle \lhd G$ and $1 \neq z \in C_{3^\infty}$ then

$$(g(x,y))^{(z,1)} = g(xz, yz^{-1})$$

must be a non-trival power of $g(x,y)$, and it clearly isn't in general. It follows that every cyclic normal subgroup of G lies in B.

For non-negative integers i, put

$$B_i = \left\{ b \in B : b^{3^i} = 1 \right\}.$$

Now B_1 contains a cyclic normal subgroup of G, namely $\langle (x, x^2) \rangle$ where x has order 3 in C_{3^∞}. Conversely, let $\langle (x, y) \rangle$ be a non-trivial cyclic normal subgroup of G. Then there is an integer i such that

$$(y^{-1}, y^{-1}x) = (x, y)^g = (x^i, y^i).$$

Thus $y^{-1} = x^i$ and $x = y^{i+1}$ so that $x^{i^2+i+1} = 1$. Now $x = 1$ implies that $y = 1$, which is not possible, so $x \neq 1$ and x has order a power of 3. In particular 3 divides $i^2 + i + 1$. Thus $i = 1 + 3k$ for some integer k (since the only solution of $i^2 + i + 1$ modulo 3 is $i = 1$). Also 9 never divides $i^2 + i + 1$, so x has order 3. Finally, $y = x^{-i} = x^{-1}x^{-3k} = x^{-1} = x^2$ so that

$$\lambda_1(G) = \big\langle (x, x^2) \big\rangle < B_1.$$

The quotient $B_1/\lambda_1(G)$ is cyclic, so $B_1 \leq \lambda_2(G)$ by 5.3.1. There is a homomorphism $G \to G$, $(u, v) \mapsto (u, v)^3$, $g \mapsto g$ for $u, v \in C_{3^\infty}$. This homomorphism is onto and has kernel B_1. Therefore $G/B_1 \cong G$. Using the same argument as above, it follows that $B_i \leq \lambda_{2i}(G)$ for all integers $i \geq 0$.

Also $\lambda_i(G) < B_i$ for all i. This has been shown for $i = 1$. Suppose this is true for $i \geq 1$. Note that $\lambda_{i+1}(G)B_i/B_i$ is generated by cyclic normal subgroups of G/B_i since $\lambda_i(G) < B_i$ and so $\lambda_{i+1}(G)B_i/B_i \leq \lambda_1(G/B_i) < B_{i+1}/B_i$ by the case $i = 1$.

Since B is the union of the B_i, we have $\lambda_\omega(G) = B$ (and $\lambda_i(G) \neq B$ for all integers $i \geq 0$). That $G = \lambda_{\omega+1}(G)$ is clear. Note that in this example, the factors of the Lambda series are abelian. □

5.3.12. Example. *The group $G = C_{3^\infty} \wr C_3$ has (abelianized) cyclic height $\omega + 1$ and $\lambda_\omega(G)$ is the base group of G.*

Proof. Any cyclic normal subgroup of G lies inside the base group B of G. Any diagonal element of B generates a cyclic normal subgroup of G. If $a \in B$ is a non-diagonal, there is $b \in B$ and $1 \neq c \in B$ with b diagonal and c with at least one trivial entry such that $a = bc$. If a generates a cyclic normal subgroup of G and g is an element of order 3 generating the top of G, there is an integer i such that

$$bc^g = a^g = a^i = b^i c^i.$$

Necessarily, $c^g c^{-i}$ is diagonal. We will show that elements of this form are not diagonal.

Without loss of generality, consider $d_i = (1, x, y)^g (1, x^i, y^i) = (y, x^i, xy^i)$. If d_i is diagonal then $y = x^i = xy^i, = x^{i^2}$, so $x^{i^2 - i + 1} = 1$. Clearly $x = 1$ if and only if $y = 1$, so suppose otherwise. The order of x is a power of 3 and 3^2 does not divide $i^2 - i + 1$. Thus x has order 3 and $y = x^i = x$ or x^2. Therefore $d_i = (x, x, x^2)$ or $(x^2, x^2, 1)$ neither of which are diagonal. It follows that $\lambda_1(G)$ is the diagonal subgroup D of G.

Now G/D is isomorphic to the group in Example 5.3.11 and if B is the base group of G, then $B/D = \lambda_\omega(G/D)$. The statement follows. □

5.3.13. Example. *There is a subgroup of* $\mathrm{GL}(5, \mathbb{C})$ *with (abelianized) cyclic height* ω. *More specifically, let*

$$a = \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 1 & 0 & 0 & & \\ & & & 1 & 0 \\ & & & 2 & 1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 1 & 0 & 0 & & \\ & & & 1 & 2 \\ & & & 0 & 1 \end{pmatrix}$$

and $c_i = \mathrm{diag}(\epsilon_i, 1, 1, 1, 1)$ *where* ϵ_i *is a* 3^i*-th primitive root of unity. Then the group* $G = \langle a, b, c_i : i = 1, 2, 3, \ldots \rangle$ *has cyclic height* ω*.*

Proof. Let

$$D = \langle \mathrm{diag}(c_i, c_j, c_k, 1, 1) : i, j, k = 1, 2, \ldots \rangle \cong C_{3^\infty} \times C_{3^\infty} \times C_{3^\infty}.$$

Then $G = D \rtimes \langle a, b \rangle$. In D, there is a series of normal subgroups of length ω of G with cyclic factors and so $D \leq \lambda_\omega(G)$ by 5.3.1.

There is a normal subgroup N of G such that DN/N is the base group in $G/N \cong C_{3^\infty} \wr C_3$ and the base group is $\lambda_\omega(G/N)$ as in 5.3.12. Now $D \leq \lambda_i(G)$ implies $DN/N \leq \lambda_i(G/N)$ for all natural numbers i by 5.3.7, and thus $D = \lambda_\omega(G)$.

Also G/D is a free group of rank 2, and so $\lambda_1(G/D) = 1$. Thus $D = \lambda_\omega(G) = \lambda(G)$. □

If one adds the condition of irreducibility on a nilpotent linear group. then the centre of the group is rather large. In particular, Suprunenko proves the following (see [39] Theorem 3.13).

5.3.14. Theorem (Suprunenko). *Let* G *be an irreducible nilpotent linear group. Then* $\zeta_1(G)$ *has finite index in* G*. Moreover, this index can be bound explicitly in terms of the degree and nilpotency class of* G*.*

Something similar happens for a supersoluble linear group G and $\lambda_1(G)$.

5.3.15. Lemma. *Let* G *be a supersoluble group containing a torsion-free abelian normal subgroup* A *of finite index in* G*. Then there is a free abelian normal subgroup of finite index in* G*, generated by cyclic normal subgroups of* G*. In particular,* $G/\lambda_1(G)$ *is finite.*

Since a completely reducible supersoluble linear group G has an abelian normal subgroup B of finite index ([39] Theorem 1.14), then Lemma 5.3.15 applies; for some finite power of B is torsion-free. Thus we have:

5.3.16. Theorem. *Let G be a supersoluble completely reducible linear group. Then $G/\lambda_1(G)$ is finite.*

Proof of Lemma 5.3.15 Regard A as a $\mathbb{Z}$-module. Put $V = \mathbb{Q} \otimes_{\mathbb{Z}} A$. The group A embeds into V via the map $a \mapsto 1 \otimes a$. We regard A and V as G-modules via conjugation.

Note that every subgroup of G is finitely generated. In particular, A is finitely generated torsion-free abelian. Let $A = \bigoplus_{i=1}^{r} \mathbb{Z}a_i$ for some $a_1, \ldots, a_r \in A$. Here $\dim_{\mathbb{Q}} V = r$.

Using the supersolubility of G, there is a G-series of A with cyclic factors. Thus there is a series of $\mathbb{Q}G$-submodules of V whose factors have $\mathbb{Q}$-dimension at most 1; for by tensoring the G-series of A by $\mathbb{Q}$, one obtains a $\mathbb{Q}G$-series of V with $\mathbb{Q}$-cyclic factors.

Now A acts trivially on V, so V is a $\mathbb{Q}\frac{G}{A}$-module. By Maschke's theorem, V is completely reducible as $\mathbb{Q}\frac{G}{A}$-module and thus as $\mathbb{Q}G$-module. Therefore, V is a direct sum of 1-dimensional $\mathbb{Q}G$-modules, $\mathbb{Q}v_i$ for $1 \leq i \leq r$.

Moreover, each

$$v_i = \sum_{j=1}^{r} \frac{m_j}{p_j} a_j$$

for some integers m_j, p_j, and thus

$$w_i = (\prod_{j=1}^{r} p_j) v_i \in A.$$

Fix $1 \leq i \leq r$ and $g \in G$. There is a rational s such that $v_i g = sv_i$. Since $0 \neq A \cap \mathbb{Q}v_i \leq_{\mathbb{Z}G} A$ and A is G-supersoluble, there is $0 \neq b \in A \cap \mathbb{Q}v_i$ such that $\mathbb{Z}b \leq_G A \cap \mathbb{Q}v_i$. Therefore, $bg = nb$ for some integer n. Since A is torsion-free and some power of g acts trivially on A, we have $n = \pm 1$. Now $b = kv_i$ some $k \in \mathbb{Q}$, so $nb = bg = s(kv_i) = sb$. Thus $s = \pm 1$. Hence for all $g \in G$ we have $w_i g = \pm w_i$ and so $\langle w_i \rangle \lhd G$.

Now $B = \langle w_i : 1 \leq i \leq r \rangle$ is a normal subgroup of G contained in A. Also $\langle w_i \rangle \leq \langle v_i \rangle$ and $V = \bigoplus_{i=1}^{r} \mathbb{Q}v_i$, so the subgroup B is free abelian of rank r. Since A is also free abelian of rank r, the quotient A/B must be finite. Thus B has finite index in G. We have exhibited a subgroup B with the required properties. Note that $B \leq \lambda_1(G)$. □

One can use Suprenenko's theorem in the theory of skew linear groups to prove the following:

5.3.17. Proposition (Zalesskii, [52]). *Let D be a locally finite-dimensional division algebra and let G be a locally nilpotent subgroup of* $\mathrm{GL}(n, D)$ *with* $\mathcal{U}(G) = 1$. *Then G is centre by locally-finite.*

A natural question, is whether one can replace "nilpotent" by "supersoluble" in 5.3.17 and conclude that $G/\lambda_1(G)$ is locally-finite.

The proof of 5.3.17 given in [34] uses the fact that the centre of a linear group is a closed subgroup. In general the subgroup $\lambda_1(G)$ is not closed in G, so a similar technique will not work.

5.3.18. Example. *Let A be a free abelian group of infinite rank and $G = A \wr C_2$. Then G is a linear group with $\lambda_1(G)$ not closed in G.*

Proof. Let $B = A \times A$ be the base group of G. Now cyclic normal subgroups of G are generated by elements of the form (a,a) or (a,a^{-1}) for $a \in A$, so

$$\lambda_1(G) = \left\langle (a,a),(a,a^{-1}) : a \in A \right\rangle .$$

Now $(a,b)^2 = (a,a^{-1})(a,a)(b,b^{-1})^{-1}(b,b)$ for all $a,b \in A$, so $B^2 \leq \lambda_1(G)$. Also $\lambda(G) = B^2 \langle (a,a) : a \in A \rangle$, since $(x,x) = (x^2,1)(x^{-1},x)$ for $x \in A$. Thus $B/\lambda_1(G)$ is an elementary abelian 2-group of infinite rank.

The base group B is linear by Mal'cev's theorem [39] 2.2. This theorem says that an abelian group Z is a linear group of degree n over some field of characteristic 0 if and only if the torsion group of Z has finite rank $\leq n$. Thus G is linear over a field F of characteristic 0.

Suppose that $\lambda_1(G)$ is closed. Then by [39] Theorem 6.4, $G/\lambda_1(G)$ is linear over F. But $G/\lambda_1(G)$ contains an elementary abelian 2-group of infinite rank, which contradicts Mal'cev's theorem. Therefore $\lambda_1(G)$ is not closed. □

We conclude with some open questions.

5. Let G be a locally supersoluble skew linear group over a locally finite-dimensional division algebra. Does G have an abelian normal subgroup A generated by cyclic normal subgroups of G such that G/A is locally finite?

6. Is there a bound on the cyclic heights of skew linear groups over locally finite-dimensional division algebras?

Chapter 6

Generalized Engel elements

It is possible to generalize the notion of nilpotence by considering commutators of the form $[x,_n g]$. This leads to "Engel theory". An aim of the theory is to determine the "Engel structure" of a group restricted by a solubility or a finiteness condition. We shall review Engel theory briefly; the reader is referred to Robinson [31] Section 12.3 for more details.

Our aim in this chapter is to provide a supersoluble analogue of Engel theory.

6.1 Engel theory

Let x, g be members of the group G. We define higher commutators $[x,_n g]$ for each positive integer n inductively:

$$[x,_1 g] = [x, g]$$

and then for $n \geq 1$,

$$[x,_{n+1} g] = [[x,_n g], g].$$

The element x is called a *right Engel* element of G if for every $g \in G$ there is a positive integer n such that

$$[x,_n g] = 1.$$

Similarly, x is called a *left Engel* element of G if for every $g \in G$ there is a positive integer n such that

$$[g,_n x] = 1.$$

Let $\underline{R}(G)$ and $\underline{L}(G)$ denote the sets of right and left Engel elements respectively. Note that it is unknown as to whether these subsets of G are actually subgroups of G.

6.1.1. Proposition. *We have the following for any group G:*

1. $\eta(G) \subseteq \underline{L}(G)$;
2. $\zeta(G) \subseteq \underline{R}(G)$;
3. *The inverse of a right Engel element is a left Engel element;*
4. $\underline{R}(G) = G$ *if and only if* $\underline{L}(G) = G$.

Part 3 of 6.1.1 is due to Heineken. A group G satisfying Part 4 of 6.1.1 is called an *Engel* group. By Part 1, any locally nilpotent group is Engel. Golod (see [8]) has found an example of a finitely generated infinite Engel p-group. This cannot be locally nilpotent; otherwise it would be finite.

If one looks at certain finiteness conditions, $\underline{L}(G)$ and $\underline{R}(G)$ are actually subgroups:

6.1.2. Theorem (Baer). *Let G be a group satisfying the maximal condition on subgroups. Then*

$$\underline{L}(G) = \eta(G) = \eta_1(G)$$

is nilpotent and

$$\underline{R}(G) = \zeta(G) = \zeta_m(G)$$

for some natural number m.

6.1.3. Corollary (Zorn). *A finite Engel group is nilpotent.*

6.1.3 was the first result to be proved about Engel groups; it is a group-theoretic version of Engel's theorem about ad-nilpotence in finite-dimensional Lie Algebras (see Humphreys [14] Section 3.2). It follows from 6.1.3 that if G is a finite group with every 2-generator subgroup nilpotent, then G is nilpotent.

Linear groups have a nice Engel structure. More precisely,

6.1.4. Theorem (Gruenberg, [9]). *Let G be a linear group. Then $\underline{L}(G) = \eta(G)$ and $\underline{R}(G) = \zeta(G)$.*

In moving up to the finitary linear and up to certain finitary skew linear cases, we lose the nice behaviour of the right Engel elements. However, the left Engel elements still behave well.

6.1.5. Theorem (Wehrfritz, [45, 43]). *Let G be a finitary skew linear group over the division ring D. Suppose that either D is a field or is a locally finite-dimensional division algebra over a perfect field. Then $\underline{L}(G) = \eta(G)$. There are examples of such groups G for which G is Engel and $\zeta(G) = 1$.*

The McLain group $M(\mathbb{Q}, D)$ is a locally nilpotent finitary group with trivial centre, giving us an example of the kind indicated in 6.1.5.

6.2 Definitions of Sengel elements

6.2.1 Definition of right Sengel elements

We shall now define a supersoluble analogue of Engel elements, as promised. Let G be a group. A *right Sengel* element of G is an $x \in G$ such that for all $g \in G$, the 2-generator group $\langle x, g \rangle$ is supersoluble and $[x, g] \in \underline{R}(G')$. We write $\underline{RS}(G)$ for the set of all right Sengel elements of G. A *right Sengel* group is one which coincides with its subset of right Sengel elements.

First we note that the subgroup $\lambda(G)$ consists of right Sengel elements:

6.2.1. Lemma. *For any group G,*

$$\lambda(G) \leq \underline{RS}(G).$$

Proof. Let $x \in \lambda(G)$ and $g \in G$. Then $\langle x, g \rangle \cap \lambda(G)$ is $\langle x, g \rangle$-hypercyclic and the quotient $\langle x, g \rangle / (\langle x, g \rangle \cap \lambda(G))$ is cyclic. Thus $\langle x, g \rangle$ is hypercyclic and so supersoluble by 1.3.1 Part 6.

Also $\lambda(G)$ is G'-hypercentral by 5.3.3, so

$$[\lambda(G), G] \leq G' \cap \lambda(G) \leq \zeta(G'), \subseteq \underline{R}(G')$$

by 6.1.1. In particular, $[x, g] \in \underline{R}(G')$. □

In our supersoluble theory, we have no suitable replacement for the commutator operation. We need something in our definition that applies to pairs of elements. The motivation for the 2-generator part of the definition comes from the following theorem, which was proved using formation theory.

6.2.2. Theorem (Carter, Fischer and Hawkes, [4]). *A finite group which has all its 2-generator subgroups supersoluble is itself supersoluble. (cf. remark after Zorn's theorem 6.1.3.)*

In a similar vein to above, Wehrfritz has shown the following result in [38]. (In the same paper, he gives a direct proof of 6.2.2.)

6.2.3. Theorem (Wehrfritz). *A linear group which has all its 2-generator subgroups supersoluble is hypercyclic.*

These theorems have an obvious consequence:

6.2.4. Corollary. *1. A finite right Sengel group is supersoluble;*

2. A linear right Sengel group is hypercyclic.

A question that remains open is the following:

6.2.5. Question. *If G is right Sengel then does it follow that G' is Engel? Certainly $\langle \underline{R}(G') \rangle = G'$.*

We conclude this section with a discussion on the strength of the definition of right Sengel element.

Obviously, for a satisfactory supersoluble Engel theory we need 6.2.4. If we remove the condition of supersolubility on the 2-generator subgroup $\langle x, g \rangle$ in the definition of right Sengel element, then Corollary 6.2.4 becomes false. For example, let G be the alternating group on 4 letters. Here, G satisfies the commutator condition $[x, g] \in \underline{R}(G') = \zeta(G') = G'$ for all $x, g \in G$, but G is not supersoluble.

Suppose that we remove the commutator condition from the definition of right Sengel element. Consider the set S of elements x of the finite group G such that $\langle x, g \rangle$ is supersoluble for all $g \in G$. Now if $S = G$ then G is supersoluble by 6.2.2. Also $\lambda(G) \subseteq S$ using the proof of 6.2.1. But there are finite groups G for which $S \neq \lambda(G)$.

For the next example we shall use the following lemma:

6.2.6. Lemma ([50] Lemma 1.3, p3). *Let p be a prime and let K be an irreducible abelian group of linear automorphisms of the finite-dimensional $\mathbb{F}_p$-vector space V. If K has exponent dividing $p-1$ then V has dimension 1.*

Let $H = \mathrm{Alt}(4)$ and let $p = 13$. Now H has a faithful irreducible linear representation of degree 3 over $\mathbb{F}_p$. So H acts on $A = \mathbb{F}_p^{(3)}$ and we can form the split extension $G = A \rtimes H$.

If $X = \langle x \rangle$ is a cyclic normal subgroup of G, then XA/A is a cyclic normal subgroup of $G/A \cong H$. Thus $x \in A$ (for $\lambda(H) = 1$). Now X is an H-submodule of A, hence it is trivial by the irreducibility of H. It follows that $\lambda(G) = 1$.

Let $x \in A$ and consider $Y = \langle x, g \rangle$ for $g \in G$. Now $A \cap Y$ is an elementary abelian p-subgroup of Y and $K = Y/(A \cap Y)$ is abelian group with exponent dividing $p - 1$ (this is because G/A has order 12). Now K acts linearly and completely reducibly (by Maschke's theorem) on $A \cap Y$. By 6.2.6 any irreducible $\mathbb{F}_p K$-module is cyclic. Thus Y is supersoluble. It follows that $A \subseteq S$, so $S \neq \lambda(G)$.

Note that the above construction works for any non-trivial finite group H such that $\lambda(H) = 1$ and with suitably chosen prime p (using Dirichlet's theorem, see [12] Theorem 15). Thus without the commutator condition we do not get a finite supersoluble version of Baer's theorem 6.1.2. We shall prove later in this chapter that our definition works for finite groups. That is, $\underline{RS}(G) = \lambda(G)$ for a finite group G.

6.2.2 Definition of left Sengel elements

One is tempted to define "left Sengel elements" as the dual of right Sengel elements, i.e. to say that x is left Sengel if for all $g \in G$ the subgroup $\langle x, g\rangle$ is supersoluble and $[g, x] \in \underline{L}(G')$. However this definition is unsatisfactory. If left Sengel elements are to be supersoluble analogues of left Engel elements then any $x \in G$ which generates a locally supersoluble normal subgroup of G ought to be left Sengel.

Let G be the alternating group on 4 symbols. Take $x = (1, 2)(3, 4)$. Then $\langle x^G \rangle$ is supersoluble but $\langle x, (1, 2, 3)\rangle = \mathrm{Alt}(4)$ is not. This example shows that the dual of right Sengel element is not the correct definition of left Sengel. Somehow we have to capture the property of normality into the definition – to do this we use conjugates.

Let x be an element of the group G. We call x *left Sengel* if for all $g \in G$ the 2-generator subgroup $\langle x^g, x\rangle$ is supersoluble and $[x^g, x] \in \underline{L}(G')$. Denote the set of all left Sengel elements of G by $\underline{LS}(G)$.

It is well-known that there is no supersoluble analogue of the Fitting subgroup or of the Hirsch-Plotkin radical. For example, by a theorem of Baer, the product of two supersoluble normal subgroups is supersoluble if and only if it is nilpotent-by-abelian (see [50] Theorem 1.13, page 8). To make the presentation of our theory similar to that of Engel theory, we introduce some notation. Let $\xi(G)$ be the union of all locally supersoluble normal subgroups of G, i.e. so that $x \in \xi(G)$ if and only if $\langle x^G \rangle$ is locally supersoluble. This will not be a subgroup of G in general. We call G *left Sengel* if G coincides with $\underline{LS}(G)$.

6.2.7. Proposition. *For any group G we have:*

$$\xi(G) \subseteq \underline{LS}(G).$$

Proof. Let $x \in \xi(G)$, so that $X = \langle x^G \rangle$ is locally supersoluble. Now X contains $\langle x^g, x\rangle$ for every $g \in G$, so each $\langle x^g, x\rangle$ is visibly supersoluble. Also X' is locally nilpotent and normal in G', so

$$[x^g, x] \in X' \leq \eta(G') \subseteq \underline{L}(G').$$

□

There is a relationship between the notions of right Sengel and left Sengel similar to that of right Engel and left Engel.

6.2.8. Proposition. *For any group G,*

$$\underline{RS}(G) \subseteq \underline{LS}(G).$$

Proof. Pick $x \in \underline{RS}(G)$ and $g \in G$. Now by definition $\langle x, x^g\rangle$ is supersoluble and $[x, x^g]$ is a right Engel element of G'. By Heineken's Theorem 6.1.1 Part 3, $[x^g, x] = [x, x^g]^{-1}$ is left Engel in G'. Therefore x is left Sengel in G. □

A consequence of Proposition 6.2.8 is that every right Sengel group is a left Sengel group. The converse is false.

6.2.9. Proposition. *There is a finite group G which is not supersoluble (in particular, not right Sengel), is the product of two normal supersoluble subgroups and is a left Sengel group.*

Proof. (cf. [50] example on page 7) Let $A = \langle x \rangle \times \langle y \rangle$ where x and y have order 5. Consider the automorphisms of A defined as follows:

$$\alpha : x \mapsto x^2, y \mapsto y^{-2};$$

$$\beta : x \mapsto y^{-1}, y \mapsto x.$$

Put $H = \langle \alpha, \beta \rangle$. The group H is Quarternion of order 8 since $|\alpha| = |\beta| = 4$ and $\alpha^\beta = \alpha^{-1}$. Let G be the obvious split extension $A \rtimes H$ and choose any $h \in H$. Then $R_h = \langle A, h \rangle$ is a proper normal subgroup of G. Also R_h is supersoluble; for R_h/A is abelian of exponent dividing 4 acting linearly on A, so any irreducible R_h-submodule of A has dimension 1 by 6.2.6. Therefore $\left\langle (ah)^G \right\rangle (\leq R_h)$ is supersoluble for all $a \in A$ and $h \in H$; that is $G = \xi(G) = \underline{LS}(G)$. Now β^2 acts non-trivially on A, so H acts faithfully and completely reducibly on A (using Maschke's theorem). Hence A is H-irreducible (otherwise, A would be the direct sum of two 1-dimensional H-modules and consequently H would have to be abelian). As a result, G is not supersoluble. However $G = R_\alpha R_\beta$. □

We shall be interested in the "Sengel structure" of groups. Our main interest is to see whether $\underline{LS}(G) = \xi(G)$ and $\underline{RS}(G) = \lambda(G)$ when G is restricted suitably. We conclude this section with a basic result.

6.2.10. Proposition. *Let N be a normal subgroup of G and let H be a subgroup of G. Then:*

1. $\underline{LS}(G) \cap H \subseteq \underline{LS}(H)$ *and* $\underline{RS}(G) \cap H \subseteq \underline{RS}(H)$*;*

2. $\underline{LS}(G)N/N \subseteq \underline{LS}(G/N)$ *and* $\underline{RS}(G)N/N \subseteq \underline{RS}(G/N)$*.*

Proof. The first part follows by restriction; if $x \in H$ is left Sengel in G, each $\langle x^h, x \rangle$ is supersoluble and $[x^h, x] \in \underline{L}(G') \cap H' \subseteq \underline{L}(H')$ for any $h \in H$. The right Sengel part is similar.

If $x, y \in G$ then $N[x, y] = [Nx, Ny]$ and $\langle Nx, Ny \rangle = N \langle x, y \rangle / N$. Thus $\langle Nx, Ny \rangle$ is supersoluble when $\langle x, y \rangle$ is supersoluble. Also $\underline{L}(G')N/N \subseteq \underline{L}(G'N/N) = \underline{L}((G/N)')$. Thus the result follows for left Sengel elements and is similar for right Sengel elements. $\square$

6.3 The Sengel structure of finite groups

In this section we show that the Sengel structure of finite groups is well-behaved. This suggests that our definitions of Sengel elements are correct, or at least reasonably good. We prove the following theorem:

6.3.1. Theorem. *Let G be a finite group. Then $\underline{RS}(G) = \lambda(G)$ and $\underline{LS}(G) = \xi(G)$.*

First, a lemma:

6.3.2. Lemma. *Let G be a finite group with unique minimal normal subgroup N such that G/N is supersoluble, $G/O_p(G)$ is a p'-group and N is a p-subgroup for some prime p. Then either G is supersoluble or $N = O_p(G)$.*

Note that $G = \mathrm{Alt}(4)$ is not supersoluble but has a unique minimal normal subgroup $V_4 = O_2(G)$ and G/V_4 is a cyclic $2'$-group.

Proof. Let $P = O_p(G)$, which contains N. Now $\zeta_1(P) \neq 1$ and is normal in G, so by uniqueness of N we have $N \leq \zeta_1(P)$, i.e. $P \leq C_G(N)$. Certainly $\{n \in N : n^p = 1\}$ is a non-trivial subgroup of N and is a normal subgroup of G, so that by minimality of N, it follows that N is an elementary abelian p-group.

Let $T = \{z \in \zeta_1(P) : z^p = 1\}, \geq N$. Now T is an elementary abelian p-group and T is an $\mathbb{F}_p(G/P)$-module via conjugation. By Maschke's theorem, T is completely reducible (G/P is a p'-group). Let Z be an $\mathbb{F}_p(G/P)$-submodule of T. Then $[Z, P] \leq [\zeta_1(P), P] = 1$ and $Z^k = Z$ for all $k \in G$. Thus $Z \lhd G$, so either $Z \geq N$ or $Z = 1$. In particular, every irreducible submodule of T contains N and so must be N. Thus $T = N$.

Suppose that $N \neq \zeta_1(P)$. Using the supersolubility of G/N we can pick $a \in \zeta_1(P) \setminus N$, with $a^p \in N$ and $M = \langle a \rangle N \lhd G$. Here M is abelian and $1 \neq \langle a^p \rangle = M^p \leq N$. Now $M^p \lhd G$, so $N = M^p$ is cyclic. But then G is supersoluble.

Assume, therefore, that $N = \zeta_1(P)$ and also that $N \neq P$. Then $N < \zeta_2(P)$, so as before we can pick $a \in \zeta_2(P) \setminus \zeta_1(P)$ with $a^p \in N$ and $M = \langle a \rangle N \lhd G$, using the supersolubility of G/N.

Now $C_P(a) = C_P(M)$ (for $a^i n = (a^i n)^z = (a^i)^z n$ if and only if $(a^i)^z = a^i$, for integer i, $n \in N$ and $z \in P$). Also $C_P(M) < P$; otherwise $a \in \zeta_1(P)$. The subgroup $C_P(M)$ is a normal subgroup of G and $N \leq M \leq C_P(M)$ because a commutes with everything in N.

Therefore there is $b \in P \setminus C_P(M)$ such that $b^p \in C_P(M)$ and $R = \langle b \rangle C_P(M) \lhd G$. We have $[M, R] = [M, \langle b \rangle]$ since $C_P(M)$ centralizes M and $[M, \langle b \rangle] = [\langle a \rangle, \langle b \rangle]$ since N centralizes P.

Clearly

$$\langle [a, b] \rangle \leq [\langle a \rangle, \langle b \rangle].$$

I claim that the reverse inclusion holds. Now $[a, b] \in [M, P] \leq [\zeta_2(P), P] \leq$

$\zeta_1(P)$, so modulo $\langle [a,b] \rangle$, the subgroups $\langle a \rangle$ and $\langle b \rangle$ commute. Thus

$$\langle [a,b] \rangle = [\langle a \rangle, \langle b \rangle],$$

so that $[M,R]$ is cyclic. Also $b \notin C_P(M)$, so $[M,R] \neq 1$.

Since $[M,R] \lhd G$, it follows that N is cyclic by minimality of N. Thus G is supersoluble.

We made the assumption that $N \neq P$ to prove this. Thus the result follows. $\square$

We now prove the first half of 6.3.1:

6.3.3. Proposition. *Let G be a finite group. Then $\underline{RS}(G) = \lambda(G)$.*

Proof. We already know that $\lambda(G) \subseteq \underline{RS}(G)$. We suppose that the result is false and choose a minimal counterexample G. Note that by Baer's theorem 6.1.2, we have $\zeta(E) = \underline{R}(E)$ for any finite group E.

1. Claim. *G has a unique minimal normal subgroup N such that N is an elementary abelian p-group for some prime p, and G' centralizes N.*

If N_1 and N_2 are distinct minimal normal subgroups of G then $N_1 \cap N_2 = 1$ and thus $G \hookrightarrow G/N_1 \times G/N_2$ via the map $x \mapsto (N_1x, N_2x)$. Let $\overline{G}$ be the image of G under this map. If $x \in \underline{RS}(G)$ then $N_ix \in \underline{RS}(G/N_i)$ by 6.2.10. The result holds for both G/N_1 and G/N_2 by hypothesis, so

$$\begin{aligned} (N_1x, N_2x) &\in (\lambda(G/N_1) \times \lambda(G/N_2)) \cap \overline{G} \\ &= \lambda(G/N_1 \times G/N_2) \cap \overline{G} \\ &\leq \lambda(\overline{G}). \end{aligned}$$

In other words $x \in \lambda(G)$ and the result holds. Thus G has a unique minimal normal subgroup N.

Now $\underline{RS}(G) \neq 1$, otherwise $\lambda(G) = \underline{RS}(G) = 1$. Let x be right Sengel in G. If $\underline{R}(G') = \zeta(G') = 1$ then $[x, g] = 1$ for all $g \in G$ and so $x \in \zeta_1(G)$. But elements of $\zeta_1(G)$ generate cyclic normal subgroups of G, so $x \in \lambda(G)$, i.e. $\underline{RS}(G) = \lambda(G)$. So $\underline{R}(G') = \zeta(G') \neq 1$ and by minimality of N we have $N \leq \zeta_1(G')$. Thus G' centralizes N and N is abelian. In the usual way, N is an elementary abelian p-group, proving Claim 1.

For this proof, we say that an element $g \in G$ acts *diagonally* on N if for some decomposition $N = \langle x_1 \rangle \times \ldots \times \langle x_n \rangle$, we have $x_i^g = x_i^{m_i}$ for some integer m_i for all $1 \leq i \leq n$, i.e. $\langle g \rangle$ acts diagonally on N as linear group.

2. Claim. *There is $g \in G$ that does not act diagonally on N.*

If otherwise, the abelian group $G/C_G(N)$ acts diagonally on N (by [39] 7.1(i)). Thus N contains a non-trivial cyclic normal subgroup of G and hence N is cyclic. Then $N \leq \lambda(G)$.

Let $x \in \underline{RS}(G)$. Then by 6.2.10, $Nx \in \underline{RS}(G/N) = \lambda(G/N)$ so $\left\langle Nx^{G/N} \right\rangle$ is G/N-supersoluble. Thus $\left\langle x^G \right\rangle$ is G-supersoluble and $\underline{RS}(G) = \lambda(G)$. This contradiction implies Claim 2.

3. Claim. *If $x \in \underline{RS}(G)$ then $N \cap \langle x, g \rangle = 1$ (with g as in Claim 2). In particular, $N \cap \underline{RS}(G) = 1$.*

If $N \cap \langle x, g \rangle \neq 1$ then there is a non-trivial cyclic normal subgroup C of $\langle x, g \rangle$ contained in N. For using the supersolubility of $\langle x, g \rangle$, there is a cyclic normal series of $\langle x, g \rangle$ through $N \cap \langle x, g \rangle$. Also C has order p.

Clearly $N = \langle C^G \rangle$. Since $G' \leq C_G(N)$, we have $C^{[h,g]} = C$ for all $h \in G$ and so $(C^h)^g = C^{gh} = C^h$. Now N is the direct product of some of the C^h, for $h \in G$, so g acts diagonally on N, contradicting Claim 2. Thus $N \cap \langle x, g \rangle = 1$ and so $N \cap \underline{RS}(G) = 1$.

4. Claim. *If $x \in \underline{RS}(G)$ then G properly contains $\langle x, g \rangle N$.*

If not, $G = \langle x, g \rangle N, = N \rtimes \langle x, g \rangle$ by Claim 3. Now $C_{\langle x,g \rangle}(N) \lhd \langle x, g \rangle$ and centralizes N, thus $N \langle x, g \rangle \leq N_G(C_{\langle x,g \rangle}(N))$, i.e. $C_{\langle x,g \rangle}(N) \lhd G$.

If $C_{\langle x,g \rangle}(N) \neq 1$ then $N \leq C_{\langle x,g \rangle}(N) \leq \langle x, g \rangle$, a contradiction. Thus $C_{\langle x,g \rangle}(N) = 1$, so that $C_G(N) = N$. (If $z \in C_G(N)$, then $z = z_1 z_2$ for some $z_1 \in N$ and $z_2 \in \langle x, g \rangle$, and for $n \in N$, we have $n^z = n^{z_2}, = n$ if and only if $z_2 = 1$, if and only if $z \in N$.)

Thus $\langle x, g \rangle \cong G/N = G/C_G(N)$ is abelian and $[x, g] = 1$. In the above argument we can replace g by gn for any $n \in N = C_G(N)$.

Hence $C_G(x) \supseteq \langle x, g, n : n \in N \rangle, = G$ so that $x \in \zeta_1(G) \leq \lambda(G)$. But then $N \leq \langle x \rangle$ and $N \cap \langle x, g \rangle = 1$, a contradiction.

5. Claim. *The following holds:*

$$\underline{RS}(G) \subseteq C_G(N).$$

Let $x \in \underline{RS}(G)$. Put $K = \langle x, g \rangle N$. By 4, $K < G$. By hypothesis $\underline{RS}(K) = \lambda(K)$. Also $x \in \underline{RS}(K)$, by Lemma 6.2.10.

If $N \cap \lambda(K) \neq 1$ then there is a cyclic normal subgroup C of K such that $C \leq N$ and C has order p. Then $C^g = C$. Using the argument of Claim 3, g acts diagonally on N. This is a contradiction.

Thus $N \cap \lambda(K) = 1$. Since both N and $\lambda(K)$ are normal subgroups of K, we have $[N, \lambda(K)] \subseteq N \cap \lambda(K) = 1$ and so $x \in \lambda(K) \subseteq C_G(N)$.

6. Claim. *We have*

$$G = \langle g \rangle \langle \underline{RS}(G) \rangle > \langle \underline{RS}(G) \rangle \geq N.$$

Also G is a soluble group and G/N is supersoluble.

Let $L = \langle g \rangle \langle \underline{RS}(G) \rangle$. Now conjugates of right Sengel elements are right Sengel, so $\langle \underline{RS}(G) \rangle \lhd G$. Since $\underline{RS}(G) \neq 1$ we have $N \leq \langle \underline{RS}(G) \rangle$.

Suppose that $L \neq G$. Then by hypothesis $\lambda(L) = \underline{RS}(L)$ and $\langle \underline{RS}(G) \rangle \leq \underline{RS}(L)$, by 6.2.10. Thus L is (L-supersoluble)-by-cyclic and so supersoluble. But then g acts diagonally on N contradicting Claim 2. Hence $L = G$.

Also $\underline{RS}(G)N/N \leq \underline{RS}(G/N) = \lambda(G/N)$, so $\langle \underline{RS}(G) \rangle /N \leq \lambda(G/N)$. Since $G/ \langle \underline{RS}(G) \rangle$ is cyclic and $\langle \underline{RS}(G) \rangle /N \leq \lambda(G/N)$ is G/N-supersoluble, it follows that G/N is supersoluble. In particular, G is soluble.

If $\langle \underline{RS}(G) \rangle = G$ then $C_G(N) = G$ by Claim 5; but then G is supersoluble. Hence $G > \langle \underline{RS}(G) \rangle$.

7. Claim. *Let $P = O_p(G)$. We have the following inclusions:*

$$N \leq G' \leq \eta_1(G) = P \leq C_G(N).$$

Since G is not abelian, we have $N \leq G'$. Trivially, $P \leq \eta_1(G)$. If $\eta_1(G)$ is not a p-group then $O_q(G) \neq 1$ for some prime $q \neq p$ and $N \not\leq O_q(G)$, a contradiction. Thus $\eta_1(G) = P$. By Claim 6 and 1.3.1, G'/N is nilpotent. Since $N \leq \zeta_1(G')$ (by Claim 1), it follows that G' is nilpotent, so $G' \leq \eta_1(G)$.

Now P is nilpotent, so $\zeta_1(P) \neq 1$ and is normal in G. Thus $N \leq \zeta_1(P)$, so $P \leq C_G(N)$.

8. Claim. *N is properly contained in P.*

If $N = P$ then G/N is abelian by 7. Then by 3, $\langle x, g\rangle$ is abelian, so $x \in C_G(g)$ for any $x \in \underline{RS}(G)$. Thus $N \leq \langle \underline{RS}(G)\rangle \leq C_G(g)$, and g acts centrally (and hence diagonally) on N, contradicting 2. This proves the claim.

9. Claim. *p is the largest prime dividing $|G|$ and G/P is a p'-group.*

Now $G \neq \langle \underline{RS}(G)\rangle$ by Claim 6. Let q be a prime dividing the order of g modulo $\langle \underline{RS}(G)\rangle$. Let $Y = \langle g^q\rangle \langle \underline{RS}(G)\rangle$. Now $Y < G$, so by minimality of G, we have $\lambda(Y) = \underline{RS}(Y) \geq \langle \underline{RS}(G)\rangle$, as in the argument of Claim 6. Since $Y/\langle \underline{RS}(G)\rangle$ is cyclic, it follows that Y is supersoluble. By the Sylow Tower Theorem ([50] Theorem 1.8, page 5), Y has a unique (normal) Sylow r-subgroup, where r is the largest prime dividing $|Y|$. Now by 6, $Y \lhd G$, so $N \leq Y$ and $p = r$. Thus one of p or q is the largest prime dividing $|G|$.

Suppose that $q > p$. Then q cannot divide $|Y|$. It follows that q^2 is not a divisor of $|G|$. By 6 and the Sylow Tower Theorem again, G/N has a unique normal cyclic subgroup $Q/N \cong C_q$.

Now $[Q/N, \langle \underline{RS}(G)\rangle /N] \subseteq Q/N \cap \langle \underline{RS}(G)\rangle /N, = 1$ (since $q \nmid |Y|$) so $[Q, \langle \underline{RS}(G)\rangle] \subseteq N$. Also $Q = N \rtimes \langle h\rangle$ where h is an element of order q in G. If h acts diagonally on N then Q is supersoluble and has a unique cyclic normal q-subgroup, which is normal in G. This is a contradiction. Thus h does not act diagonally on N. Now if $x \in \underline{RS}(G)$ then $[x, h] \subseteq [\underline{RS}(G), Q] \subseteq N$. Also by 3, $N \cap \langle x, h\rangle = 1$. Thus $[x, h] = 1$. Therefore $h \in C_G(\underline{RS}(G)) = C_G(\langle \underline{RS}(G)\rangle)$ and $N \leq \langle \underline{RS}(G)\rangle$. But then h acts diagonally on N, a contradiction. So $q \leq p$.

The supersolubility of G/N and the Sylow Tower Theorem ensure that G/P is a p'-group, giving us Claim 9. We conclude the proof using Lemma

6.3.2. By this Lemma, $N = O_p(G) = P$, contradicting Claim 8. $\square$

We now prove the left Sengel part of 6.3.1. In fact, we shall prove something stronger.

6.3.4. Proposition. *Let $G = \langle X \rangle$ be a finite group and suppose that $X^G = X$. If for all $x, y \in X$, the subgroup $\langle x, y \rangle$ is supersoluble and $[x, y] \in \underline{L}(G')$, then G is supersoluble.*

The rest of Theorem 6.3.1 follows from the Proposition. Let $x \in \underline{LS}(G)$ and put $H = \left\langle x^G \right\rangle$ (here we are taking $X = x^G = X^H$). Let $g_1, g_2 \in G$. Then $\langle x^{g_1}, x^{g_2} \rangle = \left\langle x^{g_1 g_2^{-1}}, x \right\rangle^{g_2}$ is supersoluble (since $\left\langle x^{g_1 g_2^{-1}}, x \right\rangle$ is supersoluble) and

$$[x^{g_1}, x^{g_2}] = [x^{g_1 g_2^{-1}}, x]^{g_2} \in \underline{L}(G')^{g_2} \cap H = \underline{L}(G') \cap H = \underline{L}(H').$$

Hence by the Proposition, H is supersoluble and so $\underline{LS}(G) \subseteq \xi(G)$. By 6.2.7, $\xi(G) = \underline{LS}(G)$.

Proof of the Proposition:

Suppose that the result is false and choose G to be the smallest finite group for which the result fails. Let M be a minimal normal subgroup of G. Now

$$G/M = \langle X \rangle M/M = \langle Y \rangle,$$

where $Y = \{Mx : x \in X\}$. Clearly $Y^{G/M} = Y$. Since $\langle Mx, My \rangle = \langle x, y \rangle M/M$ and $[Mx, My] = M[x, y] \in \underline{L}(G'M/M) = \underline{L}((G/M)')$ for all $x, y \in X$. The hypotheses of the Proposition hold for $G/M = \langle Y \rangle$. By minimality of G, G/M is supersoluble.

Let M_1 be minimal normal subgroup of G not equal to M. Then G/M_1 is supersoluble and $M \cap M_1 = 1$, so that $G \hookrightarrow G/M \times G/M_1$ is supersoluble. Thus M is the unique minimal normal subgroup of G.

Let $N = \eta_1(G)$. Now $N = \underline{L}(G)$ by Baer's theorem 6.1.2 and N contains $\underline{L}(G') = \eta_1(G')$. For every $x, y \in X$ we have $[x, y] \in N$. It follows that $G' \leq N$. Thus $N \neq 1$; otherwise G is abelian, a contradiction.

Since N is nilpotent and $\zeta_1(N) \lhd G$ we have $M \leq \zeta_1(N)$ and $G' \leq N \leq C_G(M)$. In the usual way, M is an elementary abelian p-group for some prime p, N is a p-group and G acts on M as an abelian group.

If every $x \in X$ acts on M diagonally, then by [39] 7.1(i), G acts on M diagonally, so that M is cyclic. But then G is supersoluble. Thus pick $x \in X$ such that x does not act diagonally on M.

Suppose that $\langle x^G \rangle \neq G$. Then $\langle x^G \rangle$ satisfies the hypothesis of the result (the set of generators here being x^G), so $\langle x^G \rangle$ is supersoluble. Also $\langle x^G \rangle \neq 1$, so $M \leq \langle x^G \rangle$. Consequently M contains a cyclic subgroup C of order p, that is normalized by $\langle x^G \rangle$. Now $C^{gx} = C^{xg}$ since G' centralizes M, and $C^{xg} = C^g$ since x normalizes C. Also $M = C^G$ is the direct product of the C^{g_i} where the g_i are distinct coset representatives of $N_G(C)$ in G. But then x acts diagonally on M, a contradiction. Thus $G = \langle x^G \rangle$.

It is sufficient to prove that $x \in \underline{RS}(G)$. For then $x \in \lambda(G)$ by 6.3.3 and $G = \lambda(G)$. Let $g \in G$. The commutator $[x, g] \in G' \leq N$ which is nilpotent, so $[x, g] \in G' = \zeta(G') = \underline{R}(G')$. Therefore, it is enough to prove that for all $g \in G$, the subgroup $H = \langle x, g \rangle$ is supersoluble.

Suppose that $H < G$. Then $\langle x^H \rangle$ is supersoluble. If $M \cap \langle x^H \rangle = 1$ then

$$H \hookrightarrow \frac{H}{H \cap M} \times \frac{H}{\langle x^H \rangle}.$$

Now HM/M is supersoluble (it is a subgroup of the supersoluble quotient G/M) and $H/\langle x^H \rangle$ is cyclic (it is generated by the image of g). Hence H is supersoluble.

If $M \cap \langle x^H \rangle \neq 1$ then there is a cyclic subgroup C of order p contained in M and normalized by $\langle x^H \rangle$. We have seen this situation twice in this section already; x must act diagonally on M so we have a contradiction.

Thus if $x \notin \underline{RS}(G)$, then $\langle x, g \rangle = G$ for some $g \in G$.

Since $G/O_p(G)$ is abelian, it must be a p'-group. Also G/M is supersoluble, so we can apply Lemma 6.3.2 to get $M = O_p(G) = N$.

Suppose $M \cap \langle x, x^g \rangle \neq 1$. Then we can choose a cyclic subgroup C in M of order p which x normalizes (using the supersolubility of $\langle x, x^g \rangle$). This implies that x acts diagonally on M, a contradiction.

Suppose that $M \cap \langle x, x^g \rangle = 1$. The group $G/M = G/N$ is abelian, so x centralizes x^g and hence $[x, g]$ also. The commutator $[x, g] \neq 1$ because G is not abelian. Therefore, x centralizes a non-trivial cyclic subgroup $\langle [x, g] \rangle$ of M (note that $G' \leq M$). Again, this ensures that x acts diagonally on M. This final contradiction completes the proof and the section. □

6.4 Sengel structure in finitely generated linear groups

In this section, we concentrate on finitely generated linear groups. In the sequel, we shall reduce the Sengel theory of certain finitary groups to the Sengel theory of finitely generated linear groups.

We shall prove results about linear groups of positive characteristic. In zero characteristic, supersolubility behaves differently. We have not investigated this case; perhaps the methods of D.Segal in [33] could be used to yield a result in this direction.

First, we deal with right Sengel elements. We need some auxiliary results.

6.4.1. Lemma. *Let $A \leq B$ be normal subgroups of G and let B/A be both finite and G/A-supersoluble. Then $G/C_G(B/A)$ is supersoluble.*

Proof. Let $H = G/C_G(B/A)$. Now H, being a group of automorphisms of B/A, is finite. So the natural split extension $K = \frac{B}{A} \rtimes H$ is a finite group.

Now B/A has a supersoluble series whose terms are normalized by G. This series is also normalized by H, and hence K. Thus $B/A \leq \lambda(K)$. Now $C_K(\lambda(K)) \leq C_K(B/A)$ so by a result of Baer (see either Hill [13] Corollary 2.9 on page 95 or [50] Theorem 7.15 on page 34), $K/C_K(B/A)$ is supersoluble.

The centralizer $C_H(B/A)$ is trivial. (If for all $a \in A$,

$$(aB)^{gC_G(B/A)} = a^g B = aB$$

then $g \in C_G(B/A)$, that is $gC_G(B/A) = C_G(B/A)$.) Thus

$$H \cong \frac{H}{C_H(B/A)} = \frac{H}{C_K(B/A) \cap H} \cong \frac{HC_K(B/A)}{C_K(B/A)} \leq \frac{K}{C_K(B/A)},$$

so H is supersoluble. □

We shall use the profinite and congruence topologies on a group of integral matrices. We discuss these notions now.

The integral general linear group $\mathrm{GL}(n, \mathbb{Z})$ of degree n can be made into a topological group using the subgroups

$$C_{\mathrm{GL}(n,\mathbb{Z})}((\mathbb{Z}/r\mathbb{Z})^{(n)}) = \mathrm{GL}(n, \mathbb{Z}) \cap (1 + r\mathrm{M}(n, \mathbb{Z})),$$

for positive integers r, as a base of open neighbourhoods of the identity. This topology is called the *congruence topology* on $\mathrm{GL}(n, \mathbb{Z})$. Naturally, one can do the same thing in the language of finitely generated free $\mathbb{Z}$-modules M; viz., the congruence topology on $\mathrm{GL}(M)$ is given by using the subgroups $C_{GL(M)}(M/rM)$, for positive integers r, as a base of open neighbourhoods of the identity.

The *profinite topology* on the group G is obtained by taking all cosets of normal subgroups of finite index in G for an open base.

We shall be interested in a small class of integral matrix groups in which these topologies turn out to be the same.

6.4.2. Theorem (Wehrfritz, [40] Theorem 2). *Let G be a soluble-by-finite subgroup of* $\mathrm{GL}(n, \mathbb{Z})$*. Then G is closed in* $\mathrm{GL}(n, \mathbb{Z})$ *in the congruence topology and the congruence topology induces the profinite topology.*

(See [40] for a proof of this, and for more on these topologies. For more general results see [42].)

Let Y be a finitely generated free abelian normal subgroup of G. Now $H = G/C_G(Y)$ acts faithfully on Y and $\mathrm{GL}(Y) \cong \mathrm{GL}(n, \mathbb{Z})$. If, further, H is soluble-by-finite then we can use the last theorem; the induced congruence topology on H is the profinite topology on H. If L is any normal subgroup of finite index in H, then L is open in the profinite topology, and hence in the congruence topology. Now L contains the identity, so it contains an induced member of the open base we specified for the congruence topology, say

$$C_H(Y/Y^i) = H \cap C_{\mathrm{GL}(Y)}(Y/Y^i)$$

for some integer i (note that we are considering Y multiplicatively). We shall use this technique below.

6.4.3. Proposition. *Let $X \lhd G$ with X polycyclic-by-finite and $X/X^m \leq \lambda(G/X^m)$ for every positive integer m. Then $X \leq \lambda(G)$.*

Proof. The number of infinite cyclic factors in any polycyclic series of any polycyclic normal subgroup of finite index of X is an invariant of X called the Hirsch number h of X. (See Scott [32] 7.1.5 for a proof of this; Scott calls polycyclic-by-finite groups, M-groups.) The proof works by induction on h.

If $h = 0$ then X is finite, so the result is obvious (take $m = |X|$ in the hypothesis). So assume that $h > 0$. Then X has a non-trivial free abelian characteristic subgroup Y ([32] 7.1.10) of rank n, say.

Now for any positive integer m, X/Y^m has a smaller Hirsch number than that of X, and $\frac{(X/Y^m)}{(X/Y^m)^r} = \frac{X}{X^r Y^m}$ is an image of X/X^r for all positive integers r. Thus X/Y^m satifies the hypothesis of the Proposition, so by induction $X/Y^m \leq \lambda(G/Y^m)$. In particular, $X/Y \leq \lambda(G/Y)$ and $Y/Y^m \leq \lambda(G/Y^m)$ for all positive integers m. It is enough to show that $Y \leq \lambda(G)$.

Let p be any prime. Then $Y/Y^p \leq \lambda(G/Y^p)$ and $G/C_G(Y/Y^p) \hookrightarrow \mathrm{GL}(n, \mathbb{F}_p)$ (for Y/Y^p is an elementary abelian p-group of rank n). By Lemma 6.4.1, $G/C_G(Y/Y^p)$ is supersoluble. Thus $G/C_G(Y/Y^p)$ is soluble of derived length $\leq 2n$ by a theorem of Huppert (see [5] Theorem 6.2A). Hence

$$G^{(2n)} \leq \bigcap_p C_G(Y/Y^p) = C_G(\frac{Y}{\bigcap_p Y^p}) = C_G(Y).$$

It follows that $H = G/C_G(Y)$ is soluble. Also H embeds into $\mathrm{GL}(Y) \cong \mathrm{GL}(n, \mathbb{Z})$. By Theorem 9.8 of [11], H is polycyclic. Therefore the natural split extension $K = Y \rtimes H$ is also polycyclic.

Let N be a normal subgroup of K of finite index r. Then $Y^r \le N$. Also $H \cap N$ has finite index in H. We apply the aforementioned technique on H (taking $L = H \cap N$). There is a positive integer s with $C_H(Y/Y^s) \le H \cap N$.

Now $Y^{rs}C_H(Y/Y^{rs})$ is normal in K; certainly Y^{rs} is normal in K, and also $[C_H(Y/Y^{rs}), Y] \le Y^{rs}$ and $[C_H(Y/Y^{rs}), H] \le C_H(Y/Y^{rs})$. Also Y/Y^{rs} is G-supersoluble, so by Lemma 6.4.1, $G/C_G(Y/Y^{rs})$ is supersoluble. Furthermore

$$\frac{H}{C_H(Y/Y^{rs})} = \frac{G/C_G(Y)}{C_{G/C_G(Y)}(Y/Y^{rs})} = \frac{G/C_G(Y)}{C_G(Y/Y^{rs})/C_G(Y)} \cong \frac{G}{C_G(Y/Y^{rs})},$$

so $H/C_H(Y/Y^{rs})$ is supersoluble.

Also $\frac{YC_H(Y/Y^{rs})}{Y^{rs}C_H(Y/Y^{rs})}$ is an image of Y/Y^{rs} and so is K-supersoluble. The quotient

$$\begin{aligned} \frac{H}{C_H(Y/Y^{rs})} = \frac{H}{(H \cap Y)C_H(Y/Y^{rs})} &= \frac{H}{H \cap YC_H(Y/Y^{rs})} \\ &\cong \frac{YH}{YC_H(Y/Y^{rs})} \\ &= \frac{K}{YC_H(Y/Y^{rs})} \end{aligned}$$

is supersoluble, so that $K/(Y^{rs}C_H(Y/Y^{rs}))$ is supersoluble and thus so too is K/N.

Therefore all finite images of K are supersoluble. By Baer's theorem (see, for example, [39] 11.11), K itself is supersoluble. In particular, $Y \le \lambda(K)$ and so $Y \le \lambda(G)$. It follows that $X \le \lambda(G)$. □

6.4.4. Corollary. *Let G be a group and $x \in \underline{RS}(G)$. If $X = \langle x^G \rangle$ is polycyclic-by-finite, then $X \le \lambda(G)$.*

Proof. For each X/X^m is both periodic and polycyclic-by-finite, so in particular is finite. Since $X^m x \in \underline{RS}(G/X^m)$, we have $X/X^m \le \lambda(G/X^m)$ by 6.4.1. The Proposition gives the result. □

6.4.5. Theorem. *Let G be a finitely generated linear group of degree n and characteristic $p > 0$. Then*

$$\underline{RS}(G) = \lambda(G).$$

Proof. Let $x \in \underline{RS}(G)$ and put $X = \langle x^G \rangle$. By Mal'cev's theorem ([39] 4.2), there is a family $\mathcal{H}$ of normal subgroups H such that each G/H is finite, is linear of degree n and characteristic p, and also $\bigcap \mathcal{H} = 1$. In particular, G is residually finite. Furthermore, if Q is a unipotent normal subgroup of G then QH/H is a unipotent normal subgroup of G/H. (Note: Since G is finitely generated, we can regard G as being "linear" over a finitely generated integral domain. Mal'cev's theorem then applies.)

Now each $XH/H = \langle Hx^{G/H} \rangle \leq \lambda(G/H)$ so in particular each XH/H is soluble of derived length at most $2n$, by Huppert's result again. Thus $X^{(2n)} \leq \bigcap \mathcal{H} = 1$, so X is soluble (of derived length $\leq 2n$).

By the Lie-Kolchin theorem [39] 5.8, the connected component X° is triangularizable. Also $X^\circ \lhd G$. Let $U = \mathcal{U}(X^\circ)$. Now X°/U is a finitely generated abelian group by [39] 4.10, hence $\langle Ux^{G/U} \rangle = X/U$ is finitely-generated-abelian by finite and thus is G/U-hypercyclic by Corollary 6.4.4.

We show that U is G-hypercyclic. Now UH/H is a unipotent normal subgroup of G/H and $UH/H \leq \lambda(G/H)$ so

$$[U,_{\frac{1}{2}n(n-1)} UG'G^{p-1}] \leq H$$

for every $H \in \mathcal{H}$ by [39] 11.12, and so $U \leq \zeta_{\frac{1}{2}n(n-1)}(UG'G^{p-1})$.

Let $U_i = U \cap \zeta_i(UG'G^{p-1})$. Now $\zeta_i(UG'G^{p-1})$ is closed in $UG'G^{p-1}$, so U_i is closed in U. Also if W_i is the closure of U_i in G, then certainly

$U_i \leq U \cap W_i$. Also $U_i = U \cap B$ for some B closed in G. Now B must contain W_i, so $U_i = U \cap B \supseteq U \cap W_i$, that is $U_i = U \cap W_i$.

Now $G/UG'G^{p-1}$ is a finitely generated periodic abelian group so is finite. Also, each U_{i+1}/U_i is centralized by $UG'G^{p-1}$. Thus if $y \in U_1$, then y has only finitely many conjugates in G. Thus $\langle y^G \rangle$ is finitely generated and hence finite. Since G is residually finite, there is a normal subgroup N of finite index in G such that $\langle y^G \rangle \cap N = 1$. As G-operator groups, $\langle y^G \rangle$ is isomorphic to $N \langle y^G \rangle / N$ which, being a subgroup of NX/N, is G-hypercyclic. Therefore $\langle y^G \rangle$ is G-hypercyclic. It follows that U_1 is G-hypercyclic.

To finish, the above method can be used again. By Theorem 6.4 of [39], there is a linear representation of G over F with kernel W_i. Hence G/W_i is a finitely generated linear group and as such is residually finite. Also

$$\frac{U_{i+1}}{U_i} = \frac{U_{i+1}}{W_i \cap U_{i+1}} \cong_G \frac{U_{i+1}W_i}{W_i}.$$

The above method can be used to prove that $U_{i+1}W_i/W_i$ is G-hypercyclic (by chosing $y \in U_{i+1}W_i/W_i$) and thus that U_{i+1}/U_i is G-hypercyclic.

Hence U, and so X, is G-hypercyclic. In particular, $x \in \lambda(G)$, finishing the proof.

□

Now we head towards the corresponding result for left Sengel elements.

6.4.6. Lemma. *Let G be a group and $x \in \underline{LS}(G)$. If $\langle x^G \rangle$ is polycyclic-by-finite, then $\langle x^G \rangle$ is supersoluble.*

Proof. Put $X = \langle x^G \rangle$. Suppose first that X is finite. Then $G/C_G(X)$, being a group of automorphisms of X, is finite. Also $C_G(X)x \in \underline{LS}(G/C_G(X)) =$

$\xi(G/C_G(X))$ by 6.2.10 and 6.3.1. Hence

$$\frac{X}{\zeta_1(X)} = \frac{X}{X \cap C_G(X)} \cong \frac{XC_G(X)}{C_G(X)}$$

is supersoluble. It follows that X is supersoluble ($\zeta_1(X)$ is X-supersoluble).

Now we look at the general case. We consider finite images of X. Let N be a normal subgroup of finite index in X. Then the core N_G of N in G is also a subgroup of finite index in X. Now $X/N_G = \langle x^{N_G} \rangle$ and so is supersoluble by above argument. The group X/N is a quotient of X/N_G, so X/N is supersoluble too. Thus all finite images of X are supersoluble. In particular, X must be polycyclic. By a theorem of Baer (see for example [39] 11.10), X is supersoluble. □

6.4.7. Theorem. *Let G be a finitely generated subgroup of* $\mathrm{GL}(n, F)$*, where F is a field of characteristic $p > 0$. Then $\underline{LS}(G) = \xi(G)$.*

Proof. Let $x \in \underline{LS}(G)$ and put $X = \langle x^G \rangle$. Using Mal'cev's theorem ([39] 4.2), there is a set $\mathcal{H}$ of normal subgroups $H \lhd G$ with $\bigcap \mathcal{H} = 1$ and each G/H isomorphic to a finite linear group of degree n and characteristic p.

Let $U = \mathcal{U}(X)$, the unipotent radical of X. If $g \in G$ then U^g is a unipotent normal subgroup of X, so $U \lhd G$. Also UH/H is unipotent in G/H (this is a detail of Mal'cev's theorem which was applied above).

Each XH/H is supersoluble; for $XH/X = \langle (Hx)^{G/H} \rangle$ and $Hx \in \underline{LS}(G/H)$, so supersolubility follows by the Sengel theory of finite groups, Theorem 6.3.1. In particular, XH/H is soluble of derived length $\leq 2n$ by Huppert's theorem ([5] Theorem 6.2A). Thus X is soluble, since $X^{(2n)} \leq \bigcap \mathcal{H} = 1$.

By the Lie-Kolchin Theorem ([39] 5.8) there is a (Zariski) closed triangularizable normal subgroup T of finite index in X. Now $\mathcal{U}(T) = U \cap T$. By [39]

Lemma 4.10, $TU/U \cong T/(U \cap T) = T/\mathcal{U}(T)$ is finitely generated abelian, so X/U is (finitely generated abelian)-by-finite. Thus X/U is supersoluble by 6.4.6.

To finish the proof, we show that U is X-hypercyclic.

Now by [39] 11.12 we have $[U,_{\frac{1}{2}n(n-1)} UX'X^{p-1}] \subseteq H$, for all $H \in \mathcal{H}$ and thus $[U,_{\frac{1}{2}n(n-1)} UX'X^{p-1}] = 1$. Let $U_i = U \cap \zeta_i(UX'X^{p-1})$, a (Zariski) closed normal subgroup in U.

Put $W_i = \overline{U_i}$, the closure of U_i in G. The subgroup U_i is closed in U, so $U_i = U \cap B$ for some closed subset B in G. Clearly $W_i \subseteq B$, so $U \cap W_i = U_i$. Each U_{i+1}/U_i is centralized by $UX'X^{p-1}$. Also $UX'X^{p-1}$ has finite index in X; for X/U is supersoluble and so $UX'X^{p-1}/U$ has finite index in X/U.

Thus if $y \in U_1$, then y has only finitely many conjugates in X. Therefore $\langle y^X \rangle$ is a finitely generated normal subgroup of X. Now U is locally finite; it is a unipotent group in characteristic p and so is a nilpotent p-group. Hence $\langle y^X \rangle$ is finite.

The group G is residually finite, so there is a normal subgroup N of finite index in G for which $N \cap \langle y^X \rangle = 1$. Hence as X-operator groups $\langle y^X \rangle$ is isomorphic to $N \langle y^X \rangle /N, \leq NX/N$, which is supersoluble, whence $\langle y^X \rangle$ is X-supersoluble. This holds for any $y \in U_1$. It follows that U_1 is X-hypercyclic.

Now by [39] 6.4, there is a linear representation of G over F with kernel W_i, for each i. Thus G/W_i is a finitely generated linear group and so is residually finite (by Mal'cev's result again).

We can use a similar argument to above to see that

$$\frac{U_{i+1}}{U_i} = \frac{U_{i+1}}{W_i \cap U_{i+1}} \cong_X \frac{U_{i+1}W_i}{W_i}$$

is X-hypercyclic.

Therefore X is hypercyclic and $x \in \xi(G)$, as required. □

6.5 Sengel structure of certain finitary skew linear groups

The aim of this section is to prove reduction theorems for certain finitary linear groups, so that the Sengel structure of these groups can be obtained from that of their finitely generated subgroups. We start by discussing right Sengel elements.

6.5.1. Example. *Let D be any division ring. Then the McLain group $G = M(\mathbb{Q}, D)$ is a right Sengel group with $\lambda(G) = 1$. In particular, $M(\mathbb{Q}, \mathbb{F}_p)$ gives us a locally finite example.*

This is analogous to the right Engel structure of finitary linear groups; the same example G is a right Engel group with trivial hypercentre. However, if we restrict ourselves to linear groups, the right Sengel structure is well-behaved. We head towards this result first.

6.5.2. Lemma. *Let V be a finite-dimensional D-G module. Suppose that for every finitely generated subgroup X of G there is a D-X series of V with 1-dimensional factors. Then there is a D-G series of V with 1-dimensional factors.*

Proof. Let $X \leq Y$ be finitely generated subgroups of G. Let U_X be the sum of the 1-dimensional D-X submodules of V. Then $0 < U_Y \leq U_X$. Choose X with $\dim_D U_X$ minimal, so now $U_Y = U_X$ for all such $Y \geq X$. We can write

$U_X = \bigoplus_{i \in I} U_{X,i}$ where $U_{X,i}$ is a non-zero D-X homogeneous component of U_X. Similarly write $U_Y = \bigoplus_{j \in J} U_{Y,j}$. For all $j \in J$ there is $i \in I$ such that $U_{Y,j} \leq U_{X,i}$. Among the X chosen above, pick X with the maximum number of non-zero components $U_{X,i}$. Then in the above we now have $U_{Y,j} = U_{X,i}$.

Let $g \in G$ and put $Z = \langle X, g \rangle$. In this case, the D-Z homogeneous components $U_{Z,k}$ must match up with the D-X homogeneous components $U_{X,i}$ by minimality. Hence $U_{X,i}$ is the direct sum of isomorphic 1-dimensional D-Z irreducibles. Thus g acts as a scalar on $U_{X,i}$ for all i. This holds for all $g \in G$, so there is a 1-dimensional D-G submodule W of V. By induction, the result holds for the D-G module V/W, thus it holds for V. □

6.5.3. Lemma. *Let A be an abelian normal subgroup of G, let X be a finitely generated subgroup of G and suppose that $A \cap Y \leq \lambda(Y)$ for every finitely generated subgroup Y of G containing X. Then A is X-hypercyclic.*

Proof. Consider $P = \lambda(AX)$. Let $y \in A$ and put $Y = \langle X, y \rangle$. Now $A \cap Y \leq \lambda(Y)$, so $A \cap Y$ is X-hypercyclic. Thus $y \in A \cap Y \leq P$. In other words we have $A \cap P = A$, as required. □

6.5.4. Proposition. *Suppose that G is a linear group of degree n with unipotent normal subgroup $U \lhd G$ such that $U \cap X \leq \lambda(X)$ for all finitely generated subgroups X of G. Then $U \leq \lambda(G)$.*

Proof. The result is clear for degree $n = 1$ or when $U = 1$, so assume otherwise. Now a unipotent normal subgroup of G is a stability group, so $W = C_V(U) \neq 0$. Put $U_1 = C_G(V/W) \cap U, \lhd G$. Now U_1 is unipotent and furthermore stabilizes the series

$$0 < W < V.$$

Now G/U_1 acts linearly on V/W, the subgroup U/U_1 is unipotent under this action, and V/W has dimension strictly less than n.

If Y/U_1 is a finitely generated subgroup of G/U_1 then $Y/U_1 = XU_1/U_1$ for some finitely generated subgroup X of G and so

$$\frac{U}{U_1} \cap \frac{XU_1}{U_1} = \frac{(U \cap X)U_1}{U_1} \leq \frac{\lambda(X)U_1}{U_1} \leq \lambda\left(\frac{Y}{U_1}\right).$$

By induction, U/U_1 is G/U_1-hypercyclic. Thus we can replace U by U_1 and show that U is G-hypercyclic to finish the proof.

Now the abelian group of homomorphisms $H = \mathrm{Hom}_F(V/W, W)$ is a G-module via the action

$$\alpha^g : v + W \mapsto (vg^{-1} + W)\alpha g,$$

where $\alpha \in H$, $g \in G$ and $v \in V$. There is a $\mathbb{Z}$-monomorphism

$$\beta : U \to H, u \mapsto (v + W \mapsto [v, u] = v(u - 1)).$$

This is a homomorphism because $[v, u_1u_2] = [v, u_1] + [vu_1, u_2]$ for $v \in V$ and $u_1, u_2 \in U$, and because $v + W = vu_1 + W$.

The group U is abelian and is a G-module by conjugation. Now if $g \in G$ and $u \in U$ then

$$\beta : u^g \mapsto (v + W \mapsto [v, u^g])$$

and

$$(u\beta)^g : v + W \mapsto (vg^{-1} + W)(u\beta)g = [vg^{-1}, u]g = [v, u^g].$$

Thus $(u\beta)^g = u^g\beta$ and β is a G-map.

Put $A = F(U\beta)$, an F-subspace of H. Then the dimension of A over F is finite. By Lemma 6.5.3, U is X-hypercyclic for every finitely generated subgroup X of G. Thus A has an FX-series whose factors have dimension at most 1 over F. Applying Lemma 6.5.2, A has such an FG-series.

If Fa is an FG-module and $\mathbb{Z}a$ is a G-module then for any $g \in G$, there is $n \in \mathbb{Z}$ with $ag = na$. Thus for every $\alpha \in F$ we have $(\alpha n)g = n(\alpha a) \in \mathbb{Z}\alpha a$. Therefore every element of Fa generates a $\mathbb{Z}$-cyclic G-module and so Fa is G-hypercyclic. Thus $U\beta$, and hence U, is G-hypercyclic, as required. □

6.5.5. Proposition. *Let G be a linear group and let $\mathcal{L}$ be the set of all finitely generated subgroups of G. Then we have*

$$\lambda(G) = \bigcup_{H \in \mathcal{L}} \bigcap_{H \leq K \in \mathcal{L}} \lambda(K).$$

Proof. We may assume that our ground field is algebraically closed. For $H \in \mathcal{L}$, put $\Lambda_H = \bigcap_{H \leq K \in \mathcal{L}} \lambda(K)$. If $H \leq L \in \mathcal{L}$ then

$$\{\lambda(K) : H \leq K \in \mathcal{L}\} \supseteq \{\lambda(K) : L \leq K \in \mathcal{L}\}$$

so that $\Lambda_H \leq \Lambda_L$. Thus $\Lambda = \bigcup_{H \in \mathcal{L}} \Lambda_H$ is a normal subgroup of G.

Also $\lambda(G) \cap H \leq \lambda(H)$ for every $H \leq G$, so

$$\lambda(G) \cap H = \lambda(G) \cap \bigcap_{H \leq K \in \mathcal{L}} K \leq \bigcap_{H \leq K \in \mathcal{L}} \lambda(K) = \Lambda_H.$$

Taking unions over all $H \in \mathcal{L}$, we get $\lambda(G) \leq \Lambda$.

Let $U = \mathcal{U}(\Lambda)$ and $X \in \mathcal{L}$. Let $y \in U \cap X$. Then $y \in \Lambda_H$ from some $H \in \mathcal{L}$ and also $y \in X$. Put $Y = \langle H, X \rangle$. Then $\Lambda_H \leq \Lambda_Y$, so $y \in \Lambda_Y \leq \lambda(Y)$. Since $\lambda(Y) \cap X \leq \lambda(X)$, we have $y \in \lambda(X)$. Hence $U \cap X \leq \lambda(X)$, for any $X \in \mathcal{L}$. By Proposition 6.5.4, $U \leq \lambda(G)$.

Now $\mathcal{U}(G) \cap \Lambda = U$ and

$$\frac{\Lambda}{U} = \frac{\Lambda}{\mathcal{U}(G) \cap \Lambda} \cong_G \frac{\Lambda \mathcal{U}(G)}{\mathcal{U}(G)} \leq \frac{G}{\mathcal{U}(G)},$$

so we may assume that G is completely reducible.

Each Λ_H, for $H \in \mathcal{L}$, is linear and hypercyclic, so in particular is soluble. The Λ_H form a local system for Λ, so Λ is soluble. We have used Zassenhaus' result which says that a locally soluble linear group is soluble ([39] Corollary 3.8).

By [39] Theorem 3.5(ii), there is an abelian normal subgroup A of G which is a subgroup of finite index in Λ. Using [39] Lemma 1.12, $G/C_G(A)$ is finite, so $G = C_G(A)X$ for some finitely generated subgroup X of G.

If $a \in A$ then there is $Y \in \mathcal{L}$ with $X \leq Y$ and $a \in \Lambda_Y$ (certainly $a \in \Lambda_H$ for some $H \in \mathcal{L}$, so take $Y = \langle H, X \rangle$ and then $a \in \Lambda_H \leq \Lambda_Y$). Now

$$\left\langle a^G \right\rangle = \left\langle a^X \right\rangle \leq \left\langle a^Y \right\rangle \leq \Lambda_Y \leq \lambda(Y).$$

Also $G = C_G(A)Y$, so any Y-hypercyclic series of $\left\langle a^G \right\rangle$ is a G-hypercyclic series of $\left\langle a^G \right\rangle$. It follows that $A \leq \lambda(G)$.

Thus we may assume that A is trivial. Then Λ is finite, so that $G/C_G(\Lambda)$ is finite. The same argument above with a chosen in Λ gives $\left\langle a^G \right\rangle \leq \lambda(G)$, and this finishes the proof. $\square$

6.5.6. Theorem. *Let G be a linear group for which each finitely generated subgroup X has the property that $\underline{RS}(X) = \lambda(X)$. Then G has this property as well, that is $\underline{RS}(G) = \lambda(G)$.*

Proof. Let $\mathcal{L}$ be the set of all finitely generated subgroups of G and suppose that $H, K \in \mathcal{L}$ with $H \leq K$. Then

$$H \cap \underline{RS}(G) \subseteq \underline{RS}(K) = \lambda(K),$$

so

$$H \cap \underline{RS}(G) \subseteq \bigcap_{H \leq K \in \mathcal{L}} \lambda(K).$$

Thus

$$\underline{RS}(G) \subseteq \bigcup_{H \in \mathcal{L}} \bigcap_{H \leq K \in \mathcal{L}} \lambda(K), = \lambda(G)$$

by Proposition 6.5.5. The reverse inclusion holds by 6.2.1. □

Using Theorem 6.5.6 and Theorem 6.4.5 gives us the result we desire:

6.5.7. Corollary. *Let G be a linear group over a field of characterisitic $p > 0$. Then*

$$\underline{RS}(G) = \lambda(G).$$

We now consider left Sengel elements, heading towards a left Sengel version of 6.5.7. However, we shall be able to prove more for left Sengel elements, namely that the left Sengel structure is well-behaved for a reasonable class of finitary skew linear groups.

6.5.8. Proposition. *Let G be a group and $\mathcal{L}$ be a local system of G. Then:*

$$\bigcup_{H \in \mathcal{L}} \bigcap_{H \leq K \in \mathcal{L}} \xi(K) = \xi(G).$$

Proof. Let x be in the left hand side of the equation. Then

$$x \in \bigcap \{\xi(K) : H \leq K \in \mathcal{L}\}$$

for some $H \in \mathcal{L}$. Let X be any finite subset of G. Now using properties of $\mathcal{L}$, we have $X \subseteq M$ for some $M \in \mathcal{L}$ and there is $K \in \mathcal{L}$ such that $\langle X, H \rangle \leq \langle M, H \rangle \leq K$. Hence $x \in \xi(K)$, so $\langle x^K \rangle$ is locally supersoluble. Now $\langle x^X \rangle$ is a finitely generated subgroup of $\langle x^K \rangle$, so $\langle x^X \rangle$ is supersoluble.

If Y is a finitely generated subgroup of $\langle x^G \rangle$ then Y lies inside some $\langle x^X \rangle$ where X is a finite subset of G. If follows that $\langle x^G \rangle$ is locally supersoluble and $x \in \xi(G)$.

Conversely, if $H \leq K$ then $\xi(H) \supseteq H \cap \xi(K)$; for if N is a locally supersoluble normal subgroup of K then $N \cap H$ is a locally supersoluble normal subgroup of H.

Let $H, K \in \mathcal{L}$ and $H \leq K$. Then

$$H \cap \xi(G) \subseteq K \cap \xi(G) \leq \xi(K),$$

so

$$H \cap \xi(G) \subseteq \bigcap \{\xi(K) : H \leq K \in \mathcal{L}\}.$$

Taking unions over $H \in \mathcal{L}$ on both sides give us the result, since

$$\begin{aligned} \bigcup_{H \in \mathcal{L}} (H \cap \xi(G)) &= \bigcup_{H \in \mathcal{L}} H \cap \xi(G) \\ &= G \cap \xi(G) = \xi(G). \end{aligned}$$

□

6.5.9. Corollary. *Let G be any group and suppose that for every finitely generated subgroup H of G we have $\xi(H) = \underline{LS}(H)$. Then*

$$\xi(G) = \underline{LS}(G).$$

Proof. Let $\mathcal{L}$ be the set of finitely generated subgroups of G. Then $\mathcal{L}$ is a local system of G. Also for any $H, K \in \mathcal{L}$, we have $H \cap \underline{LS}(G) \subseteq \underline{LS}(K) = \xi(K)$ by 6.2.10 and hypothesis. So

$$H \cap \underline{LS}(G) \subseteq \bigcap \{\xi(K) : H \leq K \in \mathcal{L}\}.$$

By Proposition 6.5.8, we have

$$\underline{LS}(G) \subseteq \bigcup_{H \in \mathcal{L}} \bigcap_{H \leq K \in \mathcal{L}} \xi(K) = \xi(G).$$

The reverse inclusion holds by 6.2.7. □

This Corollary tells us about the left Sengel structure of certain finitary skew linear groups:

6.5.10. Corollary. *Let G be a finitary skew linear group over D, a locally finite-dimensional division F-algebra, where the characteristic of D is prime. Then $\underline{LS}(G) = \xi(G)$.*

Proof. For the finitely generated subgroups of G are linear over F and F has positive characterisitic. It remains to apply 6.4.7 and 6.5.9. □

Finally, we note the following two corollaries; in the first of these the groups in question are linear of characteristic zero.

6.5.11. Corollary. *If G is polycyclic-by-finite then $\underline{RS}(G) = \lambda(G)$ and $\underline{LS}(G) = \xi(G)$.*

Proof. This follows immediately from 6.4.4 and 6.4.6. □

6.5.12. Corollary. *If G is locally (polycyclic-by-finite) then $\underline{LS}(G) = \xi(G)$. In particular, every locally finite group has this property.*

Proof. Every finitely generated subgroup of G satisfies the hypothesis by 6.5.11, so by 6.5.9 the result follows. □

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